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POSITIVE SOLUTIONS TO SINGULAR SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL POTENTIAL ON CONE–LIKE DOMAINS

Vitali Liskevich, Sofya Lyakhova, and Vitaly Moroz
School of Mathematics, University of Bristol
Bristol BS8 1TW, United Kingdom

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Abstract. We study the existence and nonexistence of positive (super-) solutions to a singular semilinear elliptic equation

$$-\nabla\cdot(|x|^A\nabla u)-B|x|^{A-2}u=C|x|^{A-\sigma}u^p$$

in cone–like domains of $\mathbb{R}^N$ $(N\geq 2)$, for the full range of parameters $A,B,\sigma,p\in\mathbb{R}$ and $C>0$. We provide a characterization of the set of $(p,\sigma)\in\mathbb{R}^2$ such that the equation has no positive (super-) solutions, depending on the values of A,B and the principal Dirichlet eigenvalue of the cross–section of the cone.

The proofs are based on the explicit construction of appropriate barriers and involve the analysis of asymptotic behavior of super-harmonic functions associated to the Laplace operator with critical potentials, Phragmén-Lindelöf type comparison arguments and an improved version of Hardy's inequality in cone–like domains.

1. Introduction and main results

We study the existence and nonexistence of positive (super-) solutions to the singular semilinear elliptic equation with critical potential

$$-\nabla\cdot(|x|^A\nabla u)-B|x|^{A-2}u=C|x|^{A-\sigma}u^p\quad\text{in }\mathcal{C}^\rho_\Omega. \tag{1.1}$$

Here $A,B\in\mathbb{R}$, $C>0$ and $(p,\sigma)\in\mathbb{R}^2$. By $\mathcal{C}^\rho_\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ we denote the cone-like domain defined by

$$\mathcal{C}^\rho_\Omega=\{(r,\omega)\in\mathbb{R}^N:\omega\in\Omega,\ r>\rho\},$$

where $\rho>0$, (r,ω) are the polar coordinates in $\mathbb{R}^N$ and $\Omega\subseteq S^{N-1}$ is a subdomain (connected open subset) of the unit sphere S^{N-1} in $\mathbb{R}^N$. Note that we do not prescribe any boundary conditions in (1.1). A nonnegative

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super-solution to (1.1) in a domain $G \subseteq \mathbb{R}^N \setminus \{0\}$ is an $0 \le u \in H^1_{loc}(G)$ such that

$$\int_G \nabla u \cdot \nabla \varphi \, |x|^A dx - B \int_G u\varphi \, |x|^{A-2} dx \ \ge \ C \int_G u^p \varphi \, |x|^{A-\sigma} dx, \qquad (1.2)$$

for all $0 \le \varphi \in H^1_c(G) \cap L^\infty_c(G)$. The notions of a nonnegative sub-solution and solution are defined similarly by replacing "$\ge$" with "$\le$" and "$=$" respectively. By the weak Harnack inequality for super-solutions any nontrivial nonnegative super-solution u to (1.1) in G is strictly positive in G, in the sense that $u^{-1} \in L^\infty_{loc}(G)$.

Equation (1.1), comprising in particular the known in astrophysics Lane-Emdem equation, is a prototype model for general semilinear equations. The qualitative theory of equations of type (1.1) has been extensively studied because of their rich mathematical structure and various applications for the whole range of the parameter $p \in \mathbb{R}$, e.g. in combustion theory ($p > 1$) [38], population dynamics ($0 < p < 1$) [32], pseudoplastic fluids ($p < 0$) [22, 28]. It has been known at least since earlier works by Serrin (cf. the references in [40]) and the celebrated paper by Gidas and Spruck [20] that equations of type (1.1) on unbounded domains admit positive (super) solutions only for specific values of $(p, \sigma) \in \mathbb{R}^2$. For instance, it is well known by now that the equation

$$-\Delta u = u^p \qquad (1.3)$$

in the exterior of a ball in $\mathbb{R}^N$ ($N \ge 3$) has no positive super-solutions if $p \le \frac{N}{N-2}$. The *critical exponent* $p^* = \frac{N}{N-2}$ is sharp in the sense that it separates the zones of existence and nonexistence, i.e. for $p > p^*$ (1.1) has positive solutions outside a ball. This result has been extended in several directions (see, e.g. [6, 8, 10, 11, 12, 13, 17, 23, 24, 25, 29, 40, 39, 37, 43, 44], references therein, and the list is by no means complete). Particularly, in [23] it was shown that the critical exponent $p^* = \frac{N}{N-2}$ is stable with respect to the change of the Laplacian by a second–order uniformly elliptic divergence type operator with measurable coefficients $-\sum \partial_i(a_{ij}\partial_j)$, perturbed by a potential, for a sufficiently wide class of potentials. For instance, for $\epsilon > 0$ the equation

$$-\Delta u - \frac{B}{|x|^{2+\epsilon}} u = u^p \qquad (1.4)$$

in the exterior of a ball in $\mathbb{R}^N$ ($N \ge 3$) has the same critical exponent as (1.3) [23, Theorem 1.2]. On the other hand it is easy to see that if $\epsilon < 0$ and $B > 0$, then (1.4) has no positive super-solutions for any $p \in \mathbb{R}$, while if $\epsilon < 0$ and $B < 0$, then (1.4) admits positive solutions for all $p \in \mathbb{R}$ ($p \ne 1$). In the borderline case $\epsilon = 0$ the critical exponent p^* becomes explicitly dependent on the parameter B. This phenomenon and its relation with Hardy type inequalities has been recently observed on a ball and/or exterior domains in [12, 17, 41] in the case $p > 1$.

The equation with first-order term

$$-\Delta u - \frac{Ax \cdot \nabla u}{|x|^{2+\epsilon}} = u^p \tag{1.5}$$

in the exterior of a ball in $\mathbb{R}^N$ $(N \geq 3)$ represents another type of behavior. If $\epsilon > 0$, then (1.5) has the same critical exponent $p^* = \frac{N}{N-2}$ as (1.3), and p^* is stable with respect to the change of the Laplacian by a second–order uniformly elliptic divergence type operator [25, Theorem 1.8]. On the other hand it is easy to see that if $\epsilon < 0$ and $A > 0$, then (1.5) has no positive super-solutions if and only if $p \leq 1$, while if $\epsilon < 0$ and $A < 0$, then (1.5) has no positive super-solutions if and only if $p \geq 1$. In the borderline case $\epsilon = 0$ the critical exponent p^* explicitly depends on the parameter A (see [37, 39] for the case $p > 1$).

When considered on cone–like domains, the nonexistence zone depends in addition on the principal Dirichlet eigenvalue of the cross-section of the cone. In the super-linear case $p > 1$ the equation

$$-\Delta u = u^p \quad \text{in } \mathcal{C}^1_\Omega \tag{1.6}$$

has been considered in [5, 6, 8] (see also [11] for systems, [26] for uniformly elliptic equations with measurable coefficients, [7, 34] for related studies of qualitative properties of solutions to semilinear equations on strips and cones). A new nonexistence phenomenon for the sublinear case $p < 1$ has been recently revealed in [27]. Particularly, it was discovered that equation (1.6) in a proper cone-like domain has two critical exponents, the second one appearing in the sublinear case, so that (1.6) has no positive super-solutions if and only if $p_* \leq p \leq p^*$, where $p_* < 1$ and $p^* > 1$. In [26] for $p > 1$ it was shown that if the Laplacian is replaced by a second–order uniformly elliptic divergence type operator $-\sum \partial_i(a_{ij}\partial_j)$, then the value of the critical exponents on the cone depends on the coefficients of the matrix $(a_{ij}(x))$ as well as on the geometry of the cross–section.

In the present paper we study equation (1.1) on cone-like domains for the full range of the parameters p, σ, A, B. Note that (1.1) can be rewritten in the form

$$-\Delta u - \frac{Ax \cdot \nabla u}{|x|^2} - \frac{B}{|x|^2}u = \frac{C}{|x|^\sigma}u^p \quad \text{in} \quad \mathcal{C}^\rho_\Omega,$$

so it represents the borderline case both with respect to the zero-order and the first-order perturbations in the linear part. As we will see below, due to the presence of the weighted function and lower order terms equation (1.1) exhibits all the cases of qualitative behavior described above for the Laplacian. Our approach to the problem in this paper is the development of the method introduced in [23] (see also [23, 25, 26, 27]) and is different from the techniques used in [5, 8, 10, 12, 17, 37, 39]. It is based on the explicit construction of appropriate barriers and involves the analysis of asymptotic behavior of super-harmonic functions associated to the Laplace operator with critical potentials, Phragmén-Lindelöf type comparison arguments and an improved version of Hardy's inequality in cone–like domains. The advantages of our approach are its transparency and flexibility. Particularly we prove the nonexistence results for the most general definition of weak solutions and avoid any assumptions on the smoothness of the boundary of the cone.

Below we denote $C_H := \frac{(2-N-A)^2}{4}$, while $\lambda_1 = \lambda_1(\Omega) \geq 0$ denotes the principal Dirichlet eigenvalue of the Laplace–Beltrami operator $-\Delta_\omega$ on Ω. First, we formulate the result in the special linear case.

Theorem 1.1. *Let $(p,\sigma) = (1,2)$. Then equation (1.1) has no positive super-solutions if and only if $B + C > C_H + \lambda_1$.*

If $B \leq C_H + \lambda_1$, then the quadratic equation

$$\gamma(\gamma + N - 2 + A) = \lambda_1 - B \tag{1.7}$$

has real roots, denoted by $\gamma^- \leq \gamma^+$. Note that if $B = C_H + \lambda_1$, then $\gamma^\pm = (2-N-A)/2$.

For $B \leq C_H + \lambda_1$ we introduce the critical line $\Lambda_*(p) = \Lambda_*(p, A, B, \Omega)$ on the (p,σ)–plane

$$\Lambda_*(p, A, B, \Omega) := \min\{\gamma^-(p-1) + 2,\ \gamma^+(p-1) + 2\} \qquad (p \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(p,\sigma) \in \mathbb{R}^2 \setminus \{1,2\} : \textit{equation (1.1) has no positive super-solutions}\}.$$

The main result of the paper reads as follows.

Theorem 1.2. *The following assertions are valid.*

(*i*) *Let $B < C_H + \lambda_1$. Then $\mathcal{N} = \{\sigma \leq \Lambda_*(p)\}$.*

(*ii*) *Let $B = C_H + \lambda_1$. Then*

$$\{\sigma < \Lambda_*(p)\} \cup \{\sigma = \Lambda_*(p),\ p \geq -1\} \subseteq \mathcal{N} \subseteq \{\sigma \leq \Lambda_*(p)\}.$$

If $\Omega = S^{N-1}$, then $\mathcal{N} = \{\sigma < \Lambda_*(p)\} \cup \{\sigma = \Lambda_*(p),\ p \geq -1\}$.

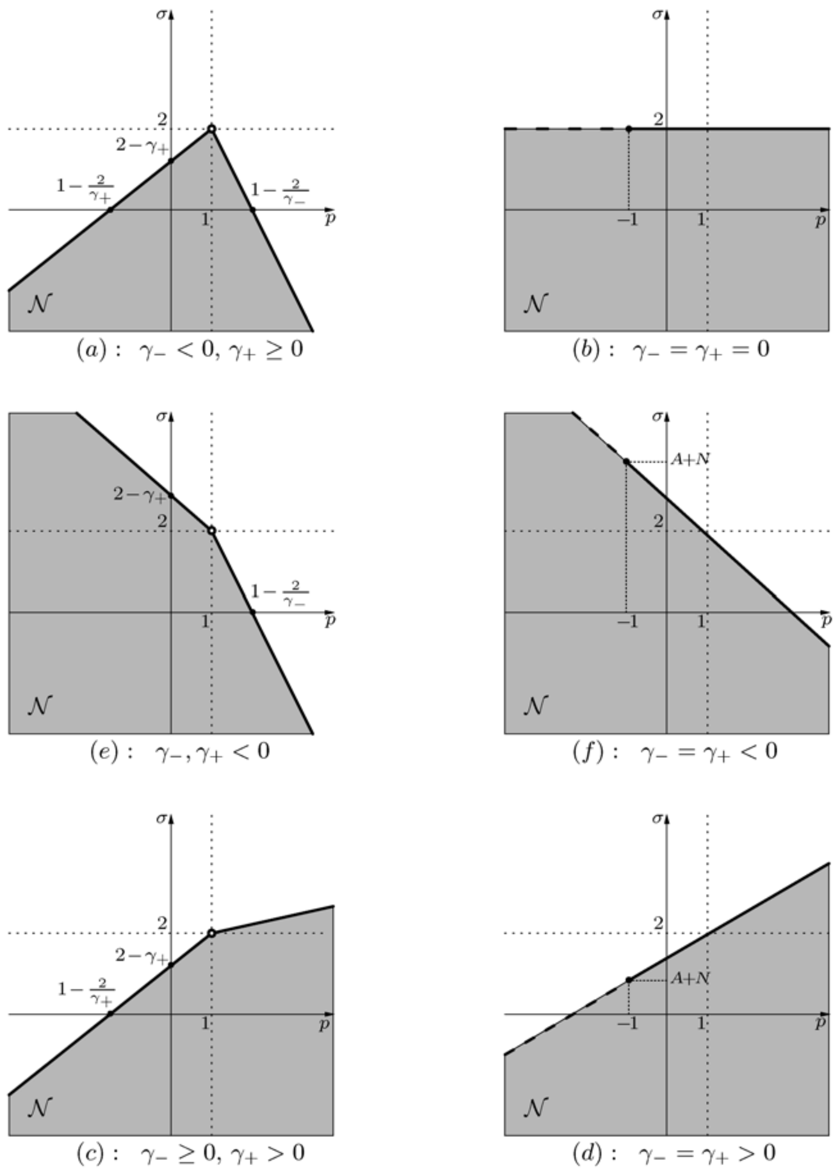

FIGURE 1. The nonexistence set $\mathcal{N}$ of equation (1.1) for typical values of γ^- and γ^+.

Remark 1.3. (i) Observe that the nonexistence set $\mathcal{N}$ does not depend on the value of the parameter $C > 0$ in (1.1). In view of the scaling invariance of (1.1) the set $\mathcal{N}$ also does not depend on the value of $\rho > 0$.
(ii) Using sub and super-solutions techniques one can show that if (1.1) has a positive super-solution in $\mathcal{C}^\rho_\Omega$, then it has a positive solution in $\mathcal{C}^\rho_\Omega$ (cf. [26, Proposition 1.1]). Thus for any $(p,\sigma) \in \mathbb{R}^2 \setminus \mathcal{N}$ equation (1.1) admits positive solutions.
(iii) In the case of proper domains $\Omega \Subset S^{N-1}$, the existence (or nonexistence) of positive super-solutions to (2.1) with $B = C_H + \lambda_1$, $p < -1$ and $\sigma = \Lambda_*(p)$ becomes a more involved issue that remains open at the moment. We will return to this problem elsewhere.

Remark 1.4. Figure 1 shows the qualitative pictures of the set $\mathcal{N}$ for typical values of γ^-, γ^+. The case (a) is typical for $A, B = 0$. The case (b) occurs, e.g., when $A, B = 0$ and $N = 2$. The cases (c) and (d) appear, in particular, when $A = 0$ ($B < C_H$ and $B = C_H$ respectively). The cases (e) and (f) are never realized by (1.1) with $A = 0$. Assume, for instance, that $B = 0$, $\lambda_1 = 0$ and $\sigma = 0$. Then (1.1) admits at most one critical exponent p^* which, depending whether $N + A > 2$ or $N + A < 2$, appears in the superlinear case ($p > 1$) or sublinear case ($p < 1$). In the former case there are no positive super-solutions if and only if $p \le p^*$, whereas in the latter if and only if $p^* \le p$. Thus $N + A$ plays a role of the “effective dimension”. Similar behavior is exhibited by second-order elliptic nondivergent type equations with measurable coefficients $-\sum a_{ij}\partial^2_{ij} u = u^p$ in the exterior of a ball in $\mathbb{R}^N$, which were recently studied in [24]. The value of the critical exponent for such equations depends on the behavior of the matrix $(a_{ij}(x))$ at infinity, though not directly but via an “effective dimension” which is determined by the asymptotic of $(a_{ij}(x))$.

Applying the Kelvin transformation $y = y(x) = \frac{x}{|x|^2}$ one sees that if u is a positive (super) solution to equation (1.1), then $\check{u}(y) = |y|^{2-N} u(x(y))$ is a positive (super) solution to the equation

$$-\nabla \cdot (|x|^A \nabla \check{u}) - B|x|^{A-2}\check{u} = C|x|^{A-s}\check{u}^p \quad \text{in} \quad \check{\mathcal{C}}^1_\Omega, \tag{1.8}$$

where $s = (N+2) - p(N-2) - \sigma$ and $\check{\mathcal{C}}^1_\Omega := \{(r,\omega) \in \mathbb{R}^N : \omega \in \Omega,\ 0 < r < 1\}$ is the interior cone–like domain. For equation (1.8) we define the critical line

$$\Lambda^*(p, A, B, \Omega) := \max\{\gamma^-(p-1) + 2,\ \gamma^+(p-1) + 2\} \qquad (p \in \mathbb{R}),$$

and the set $\check{\mathcal{N}} = \{(p,s) \in \mathbb{R}^2 \setminus \{1,2\} : (1.8)$ *has no positive super-solutions*$\}$. The following theorem extends the results in [41, 12, 17] ($A = 0$) and [39, 37]

($B = 0$), obtained on the punctured ball in the super-linear case $p > 1$. It is derived from Theorem 1.2 via the Kelvin transformation.

Theorem 1.5. *The following assertions are valid.*

(*i*) *Let* $B < C_H + \lambda_1$. *Then* $\mathcal{N} = \{s \geq \Lambda^*(p)\}$.

(*ii*) *Let* $B = C_H + \lambda_1$. *Then*

$$\{s > \Lambda^*(p)\} \cup \{s = \Lambda^*(p),\, p \geq -1\} \subseteq \mathcal{N} \subseteq \{s \geq \Lambda^*(p)\}.$$

If $\Omega = S^{N-1}$, *then* $\mathcal{N} = \{s > \Lambda^*(p)\} \cup \{s = \Lambda^*(p),\, p \geq -1\}$

Theorem 1.2 is proved in the paper after a reduction of (1.1) to the uniformly elliptic case $A = 0$. This reduction is described in Section 2 below. The rest of the paper is organized as follows. In Section 3 we prove a version of the improved Hardy inequality in cone–like domains. In Section 4 we study asymptotical behavior of super-solutions to certain linear equations. The proof of the main result is contained in Section 5 (super-linear case $p \geq 1$) and Section 6 (sub-linear case $p < 1$). Finally, Appendix includes auxiliary results on the relation between the existence of positive solutions to linear equations and positivity properties of the corresponding quadratic forms.

2. Equivalent statement of the problem

The next lemma shows that a simple transformation allows one to reduce equation (1.1) to the uniformly elliptic case $A = 0$.

Lemma 2.1. *The function u is a (super-) solution to equation* (1.1) *if and only if* $w(x) = |x|^{\frac{A}{2}} u(x)$ *is a (super-) solution to the equation*

$$-\Delta w - \frac{\mu}{|x|^2} w = \frac{C}{|x|^s} w^p \quad \text{in } \mathcal{C}_\Omega^\rho, \tag{2.1}$$

where $\mu = B - \frac{A}{2}(\frac{A}{2} + N - 2)$ *and* $s = \sigma + \frac{A}{2}(p-1)$.

Proof. This follows from a direct computation. □

The existence of positive solutions to (2.1) is intimately related to an associated Hardy type inequality for exterior cone–like domains, which has the form

$$\int_{\mathcal{C}_\Omega^\rho} |\nabla u|^2 dx \geq (C_H + \lambda_1) \int_{\mathcal{C}_\Omega^\rho} \frac{u^2}{|x|^2} dx, \qquad \forall u \in C_c^\infty(\mathcal{C}_\Omega^\rho), \tag{2.2}$$

where $C_H := \frac{(N-2)^2}{4}$ and the constant $C_H + \lambda_1$ is sharp. We prove a refined version of (2.2) in Section 3. By virtue of Lemma A.9 in Appendix, inequality (2.2) implies that equation (2.1) has positive super-solutions if and only if $\mu \leq C_H + \lambda_1$, see Remark 3.3 below. Theorem 1.1 is an immediate consequence of this result.

If $\mu \leq C_H + \lambda_1$, then the quadratic equation

$$\alpha(\alpha + N - 2) = \lambda_1 - \mu. \tag{2.3}$$

has real roots, denoted by $\alpha^- \leq \alpha^+$. If $\mu = C_H + \lambda_1$, then $\alpha^\pm = \frac{2-N}{2}$. In this case we write $\alpha_* := \frac{2-N}{2}$ for convenience. As before, we introduce the critical line

$$\Lambda(p) = \Lambda(p, \mu, \Omega) := \min\{\alpha^-(p-1) + 2,\ \alpha^+(p-1) + 2\} \qquad (p \in \mathbb{R}),$$

and the nonexistence set

$$\mathcal{N} = \{(p, s) \in \mathbb{R}^2 \setminus \{1, 2\} : \textit{equation (2.1) has no positive super-solutions}\}.$$

Theorem 1.2 is a direct consequence of the next result.

Theorem 2.2. *The following assertions are valid:*

(*i*) *Let* $\mu < C_H + \lambda_1$. *Then* $\mathcal{N} = \{\sigma \leq \Lambda(p)\}$.

(*ii*) *Let* $\mu = C_H + \lambda_1$. *Then*

$$\{\sigma < \Lambda(p)\} \cup \{\sigma = \Lambda(p),\ p \geq -1\} \subseteq \mathcal{N} \subseteq \{\sigma \leq \Lambda(p)\}.$$

If $\Omega = S^{N-1}$, *then* $\mathcal{N} = \{\sigma < \Lambda(p)\} \cup \{\sigma = \Lambda(p),\ p \geq -1\}$.

Observe that in view of the scaling invariance of (2.1) if $w(x)$ is a solution to (2.1) in $\mathcal{C}_\Omega^\rho$, then $\tau^{\frac{2-s}{p-1}} w(\tau y)$ is a solution to (2.1) in $\mathcal{C}_\Omega^{\rho/\tau}$, for any $\tau > 0$. So in what follows, for $p \neq 1$, we confine ourselves to the study of solutions to (2.1) on $\mathcal{C}_\Omega^1$. For the same reason, for $p \neq 1$ we may assume that $C = 1$, when convenient. In the remaining part of the paper we prove Theorem 2.2.

3. Improved Hardy inequality on cone–like domains

Here and thereafter, for $0 \leq \rho < R \leq +\infty$, we denote

$$\mathcal{C}_\Omega^{(\rho,R)} := \{(r, \omega) \in \mathbb{R}^N : \omega \in \Omega,\ r \in (\rho, R)\}, \quad \mathcal{C}_\Omega^\rho := \mathcal{C}_\Omega^{(\rho,+\infty)}, \quad \mathcal{C}_\Omega = \mathcal{C}_\Omega^0,$$

where $\Omega \subseteq S^{N-1}$ is a subdomain of $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$. We write $\Omega' \Subset \Omega$ if Ω' is a smooth proper subdomain of Ω such that $\Omega' \neq \Omega$ and $cl\,\Omega' \subset \Omega$. By $c, c_1, \ldots$ we denote various positive constants exact values of which are irrelevant. For two positive functions φ_1 and φ_2 we write $\varphi_1 \asymp \varphi_2$ if there exists a constant $c \geq 1$ such that $c^{-1}\varphi_1 \leq \varphi_2 \leq c\varphi_1$.

Consider the linear equation

$$-\Delta v - \frac{V(\omega)}{|x|^2} v = 0 \quad \text{in} \quad \mathcal{C}_\Omega^\rho, \tag{3.1}$$

where $\Omega \subseteq S^{N-1}$ ($N \geq 2$) is a domain, $V \in L^\infty(\Omega)$ and $\rho \geq 1$. Recall that in the polar coordinates (r, ω) the operator $-\Delta - \frac{V(\omega)}{|x|^2}$ has the form

$$-\frac{\partial^2}{\partial r^2} - \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left\{-\Delta_\omega - V(\omega)\right\},$$

where Δ_ω denotes the Dirichlet Laplace–Beltrami operator on Ω. In what follows, $\lambda_{1,V}$ denotes the principal eigenvalue of the operator $-\Delta_\omega - V$ on Ω.

The existence of positive solutions to (3.1) is equivalent to the positivity of the quadratic form

$$\mathcal{E}_V(v,v) := \int_{\mathcal{C}_\Omega^\rho} \left(|\nabla v|^2 - \frac{V(\omega)}{|x|^2} v^2\right) dx \qquad (v \in H_c^1(\mathcal{C}_\Omega^\rho) \cap L_c^\infty(\mathcal{C}_\Omega^\rho)),$$

that corresponds to (3.1), see [2] or Lemma A.9 in Appendix. Below we establish an improved Hardy type inequality on cone–like domains, which is appropriate for our purposes. Similar inequalities were obtained recently on the ball and exterior domains in [1, 18].

Theorem 3.1. *The following inequality holds:*

$$\mathcal{E}_V(v,v) \geq (C_H + \lambda_{V,1}) \int_{\mathcal{C}_\Omega^\rho} \frac{v^2}{|x|^2} dx + \frac{1}{4} \int_{\mathcal{C}_\Omega^\rho} \frac{v^2}{|x|^2 \log^2 |x|} dx \tag{3.2}$$

for all $v \in H_c^1(\mathcal{C}_\Omega^\rho) \cap L^\infty(\mathcal{C}_\Omega^\rho)$, where $C_H := \left(\frac{N-2}{2}\right)^2$. The constants $C_H + \lambda_{V,1}$ and $\frac{1}{4}$ are optimal in the sense that the inequality

$$\mathcal{E}_V(v,v) \geq \mu \int_{\mathcal{C}_\Omega^\rho} \frac{v^2}{|x|^2} dx + \epsilon \int_{\mathcal{C}_\Omega^\rho} \frac{v^2}{|x|^2 \log^2 |x|} dx \tag{3.3}$$

fails on a function $v \in H_c^1(\mathcal{C}_\Omega^\rho) \cap L^\infty(\mathcal{C}_\Omega^\rho)$ in any of the following two cases:

i) $\mu = C_H + \lambda_{V,1}$ and $\epsilon > 1/4$,

ii) $\mu > C_H + \lambda_{V,1}$ and $\forall\, \epsilon \in \mathbb{R}$.

Proof. Let $\phi_*(r,\omega) = r^{\alpha_*} \log^{\frac{1}{2}}(r) \phi_{V,1}(\omega)$, where $\alpha_* = \frac{2-N}{2}$ and $\phi_{1,V} > 0$ is the eigenfunction of $-\Delta_\omega - V$, that corresponds to $\lambda_{1,V}$. A direct computation shows that $\phi_* \in H_{loc}^1(\mathcal{C}_\Omega^\rho)$ solves the equation

$$-\Delta v - \frac{V(\omega)}{|x|^2} v - \frac{C_H + \lambda_1}{|x|^2} v - \frac{1/4}{|x|^2 \log^2 |x|} v = 0 \quad \text{in} \;\; \mathcal{C}_\Omega^\rho. \tag{3.4}$$

Thus the validity of (3.2) follows from Lemma A.9.

Now we verify the optimality of constants in (3.2). Fix $\rho \geq 1$ and let $R \gg \rho$. Similarly to [1], define a Lipschitz cut–off function

$$\theta_{\rho,R}(r) := \begin{cases} 0 & \text{for } r \leq \rho \text{ and } r > R^2, \\ r-\rho & \text{for } \rho < r \leq \rho+1, \\ 1 & \text{for } \rho+1 < r \leq R, \\ \frac{\log(R^2/r)}{\log(R)} & \text{for } R < r \leq R^2. \end{cases} \tag{3.5}$$

(i) We show that (3.3) with $\mu = C_H + \lambda_{V,1}$ and $\epsilon > 1/4$ fails on functions $\phi_*\theta_{\rho,R} \in H^1_c(\mathcal{C}^\rho_\Omega) \cap L^\infty(\mathcal{C}^\rho_\Omega)$. By Lemma A.9 direct computations give

$$\begin{aligned}
&\mathcal{E}_V(\phi_*\theta_{\rho,R}, \phi_*\theta_{\rho,R}) - (C_H+\lambda_1)\int_{\mathcal{C}^\rho_\Omega} \frac{\phi_*^2\theta^2_{\rho,R}}{|x|^2}dx - \epsilon\int_{\mathcal{C}^\rho_\Omega}\frac{\phi_*^2\theta^2_{\rho,R}}{|x|^2\log^2|x|}dx \\
&= \int_\rho^\infty\int_\Omega \Big|\nabla\Big(\frac{\phi_*^2\theta^2_{\rho,R}}{\phi_*}\Big)\Big|^2\phi_*^2\,d\omega\, r^{N-1}dr \\
&-\Big(\epsilon-\frac{1}{4}\Big)\int_\rho^\infty\int_\Omega\frac{\phi_*^2\theta^2_{\rho,R}}{r^2\log^2(r)}\,d\omega\, r^{N-1}\,dr \\
&= \int_\rho^{R^2}|\nabla\theta_{\rho,R}(r)|^2\, r\log(r)\,dr - \Big(\epsilon-\frac{1}{4}\Big)\int_\rho^{R^2}\frac{\theta^2_{\rho,R}(r)}{r\log(r)}\,dr \\
&\leq c_1 + \int_R^{R^2}\frac{\log(r)}{r\log^2(R)}\,dr - \Big(\epsilon-\frac{1}{4}\Big)\int_{\rho+1}^R\frac{1}{r\,\log r}\,dr \\
&\leq c_2 - \Big(\epsilon-\frac{1}{4}\Big)\log\log(R) \to -\infty \quad \text{as } \ R\to\infty.
\end{aligned}$$

Observe that the result does not depend on the initial choice of $\rho \geq 1$.

(ii) Choosing $\phi_*(\rho,\omega) = r^{\alpha_*}\phi_{V,1}(\omega)$, one can verify that (3.3) with $\mu > C_H + \lambda_{V,1}$ and any $\epsilon \in \mathbb{R}$ fails on functions $\theta_{\rho,R}\phi_* \in H^1_c(\mathcal{C}^\rho_\Omega) \cap L^\infty(\mathcal{C}^\rho_\Omega)$ for large $R \gg \rho$. We omit the details. □

The optimality of the improved Hardy inequality (3.2), via Corollary A.10, implies the following nonexistence result, which is one of the main tools in our proofs of nonexistence of positive solutions to semilinear equation (2.1).

Corollary 3.2. *Equation* (3.1) *has a positive super-solution if and only if* $C_H + \lambda_{V,1} \geq 0$.

Remark 3.3. In particular, if $V(\omega) \equiv \mu$, then equation (3.1) has a positive super-solution if and only if $\mu \leq C_H + \lambda_1$.

4. ASYMPTOTICS OF POSITIVE SUPER-SOLUTIONS TO $-\Delta v - \frac{V(\omega)}{|x|^2}v = 0$

According to Corollary 3.2, equation (3.1) admits positive super-solutions if and only if $C_H + \lambda_{V,1} \geq 0$. In this section, by constructing appropriate comparison functions, we obtain sharp two-sided bounds on the growth at infinity of super-solutions to (3.1).

Consider the Laplace–Beltrami operator $-\Delta_\omega - V$, where $\Omega \subseteq S^{N-1}$ ($N \geq 2$) is a domain and $V \in L^\infty(\Omega)$. Throughout this section $(\lambda_{V,k})_{k\in\mathbb{N}}$ denotes the sequence of Dirichlet eigenvalues of $-\Delta_\omega - V$ in $L^2(\Omega)$,

$$\lambda_{V,1} < \lambda_{V,2} \leq \cdots \leq \lambda_{V,k} \leq \ldots.$$

By $(\phi_{V,k})_{k\in\mathbb{N}}$ we denote the corresponding orthonormal basis of eigenfunctions in $L^2(\Omega)$, with the positive principal eigenfunction $\phi_{V,1} > 0$. If $V = 0$ and there is no ambiguity we simply write λ_k and ϕ_k.

Let $\psi \in L^2(\Omega)$. Then

$$\psi = \sum_{k=1}^{\infty} \psi_k \phi_{V,k}, \quad \text{where} \quad \psi_k = \int_\Omega \psi \phi_{V,k}\, d\omega, \tag{4.1}$$

and the series converges in $L^2(\Omega)$ with $\|\psi\|_{L^2}^2 = \sum_{k=1}^\infty \psi_k^2$. If, in addition, $\psi \in H_0^1(\Omega)$, then (4.1) converges in $H_0^1(\Omega)$ with $\|\nabla\psi\|_{L^2}^2 \asymp \sum_{k=1}^\infty \lambda_{V,k}\psi_k^2$. In what follows we use the following simple observation.

Lemma 4.1. *Let $\psi \in C_c^\infty(\Omega)$. Then the Fourier series* (4.1) *converges in $L^\infty(\Omega)$.*

Proof. Observe that $\|\phi_k\|_{L^\infty} \leq c\lambda_{V,k}^{\frac{N-1}{4}}$, by the standard elliptic estimates of eigenfunctions of the Dirichlet Laplace–Beltrami operator $-\Delta_\omega - V$ on $\Omega \subseteq S^{N-1}$, see, e.g. [14, p.172]. Choose $b > (N-1)/2$ and $a = (N-1)/4 - b$. Then

$$\sum_{k=1}^\infty |\psi_k||\phi_{V,k}| \leq c\sum_{k=1}^\infty |\psi_k|\lambda_{V,k}^{\frac{N-1}{4}} \leq c\Big(\sum_{k=1}^\infty \lambda_{V,k}^{2a}\Big)^{1/2}\Big(\sum_{k=1}^\infty |\psi_k|^2\lambda_{V,k}^{2b}\Big)^{1/2} < \infty.$$

Here the first series converges due to the classical spectral asymptotics $\lambda_{V,k} \asymp k^{\frac{2}{N-1}}$. The second series converges by a standard spectral argument, taking into account that $\psi \in \cap_{m\in\mathbb{N}} D((-\Delta_\omega)^m)$, where $D((-\Delta_\omega)^m)$ is the domain of the m–th power of the Dirichlet Laplace–Beltrami operator $-\Delta_\omega$ on Ω. $\square$

If $C_H + \lambda_{V,1} \geq 0$, then the roots of the quadratic equation

$$\alpha(\alpha + N - 2) = \lambda_{V,k} \tag{4.2}$$

are real, for each $k \in \mathbb{N}$. Denote these roots by $\alpha^-_{V,k} \le \alpha^+_{V,k}$. If $C_H + \lambda_{V,1} = 0$ and $k = 1$, then (4.2) has the unique root, denoted by $\alpha_* := \alpha^\pm_{V,1} = \frac{2-N}{2}$.

For a positive function $u \in H^1_{loc}(\mathcal{C}^1_\Omega)$ and a subdomain $\Omega' \subseteq \Omega$, denote

$$m_u(R,\Omega') := \inf_{\mathcal{C}^{(R/2,R)}_{\Omega'}} u, \qquad M_u(R,\Omega') := \sup_{\mathcal{C}^{(R/2,R)}_{\Omega'}} u.$$

Our main result in this section reads as follows.

Theorem 4.2. *Let $u \in H^1_{loc}(\mathcal{C}^1_\Omega)$ be a positive super-solution to* (3.1). *Then for every proper subdomain $\Omega' \Subset \Omega$ and $R \gg 1$ the following hold:*

(i) if $C_H + \lambda_{V,1} > 0$, then

$$c_1 R^{\alpha^-_{V,1}} \le m_u(R,\Omega') \le c_2 R^{\alpha^+_{V,1}}, \tag{4.3}$$

(ii) if $C_H + \lambda_{V,1} = 0$, then

$$c_1 R^{\alpha_*} \le m_u(R,\Omega') \le c_2 R^{\alpha_*} \log(R). \tag{4.4}$$

Remark 4.3. The above estimates are sharp, as one sees comparing with the explicit solutions $r^{\alpha^\pm_{V,1}}\phi_{V,1}$ in the case (i) and $r^{\alpha_*}\phi_{V,1}$ and $r^{\alpha_*}\log(r)\phi_{V,1}$ in the case (ii).

Remark 4.4. Equation (3.1) is invariant with respect to scaling. Namely, if $v(x)$ is a (super-) solution to (3.1) in $\mathcal{C}^\rho_\Omega$, then $v(\tau x)$ is a solution to (3.1) in $\mathcal{C}^{\tau\rho}_\Omega$, for any $\tau > 0$. Therefore, in what follows we may consider (3.1) in $\mathcal{C}^\rho_\Omega$ with a conveniently fixed radius $\rho \ge 1$.

Remark 4.5. The scaling invariance implies that positive (super-) solutions to (3.1) satisfy the Harnack inequalities with r–independent constants. More precisely, if $u > 0$ is a super-solution to (3.1), then the weak Harnack inequality reads as

$$\int_{\mathcal{C}^{(R/2,R)}_{\Omega'}} u\,dx \le C_w\, R^N m_u(R,\Omega'), \tag{4.5}$$

where $C_w = C_w(\Omega') > 0$ does not depend on $R \gg 1$. Similarly, if $u > 0$ is a solution to (3.1), then by the strong Harnack inequality

$$M_u(R,\Omega') \le C_s\, m_u(R,\Omega'), \tag{4.6}$$

where $C_s = C_s(\Omega') > 0$ is independent of $R \gg 1$.

In the remaining part of the section we prove Theorem 4.2. Our proof relies on the comparison principle in the extended Dirichlet spaces associated to (3.1) (see Appendix A). The cases (i) and (ii) are considered separately.

4.1. **Case $C_H + \lambda_{V,1} > 0$.** In this case the Hardy inequality (3.2) implies that the form $\mathcal{E}_V$ satisfies λ–property (A.5) with $\lambda(x) = \frac{C_H+\lambda_{V,1}}{|x|^2}$. Hence the extended Dirichlet space $\mathcal{D}(\mathcal{E}_V, \mathcal{C}^2_\Omega)$ is well defined (see Appendix A) and the comparison principle (Lemma A.8) is valid. Moreover, Hardy's inequality (3.2) implies that

$$c_1 \int_{\mathcal{C}^2_\Omega} |\nabla u|^2\, dx \le \mathcal{E}_V(u,u) \le c_2 \int_{\mathcal{C}^2_\Omega} |\nabla u|^2\, dx, \qquad \forall\, u \in C^\infty_c(\mathcal{C}^2_\Omega),$$

where $c_1 = 1 - \left(1 + \frac{C_H+\lambda_{V,1}}{\|V^+\|_{L^\infty}}\right)^{-1}$ and $c_2 = 1 + \frac{\|V^-\|_{L^\infty}}{C_H+\lambda_{V,1}}$. Therefore,

$$\mathcal{D}(\mathcal{E}_V, \mathcal{C}^2_\Omega) = D^1_0(\mathcal{C}^2_\Omega),$$

where $D^1_0(\mathcal{C}^2_\Omega)$ is the usual homogeneous Sobolev space, defined as the completion of $C^\infty_c(\mathcal{C}^2_\Omega)$ with respect to the Dirichlet norm $\|\nabla u\|_{L^2}$.

Lower estimate. Fix a smooth proper subdomain $\Omega' \Subset \Omega$ and a function $0 \lneq \psi \in C^\infty_c(\Omega')$. For $(r,\omega) \in \mathcal{C}^2_\Omega$ and $k \in \mathbb{N}$ set

$$v_k(r,\omega) := \eta_k(r)\phi_{V,k}(\omega), \quad \text{where} \quad \eta_k(r) := \left(\frac{r}{2}\right)^{\alpha^-_{V,k}}. \tag{4.7}$$

Define the *comparison function* v_ψ by

$$v_\psi := \sum_{k=1}^\infty \psi_k v_k, \tag{4.8}$$

where ψ_k are the Fourier coefficients of ψ as in (4.1). Thus, $v_\psi(2,\omega) = \psi(\omega)$. A direct computation verifies that $v_\psi \in H^1_{loc}(\mathcal{C}^2_\Omega)$ is a solution to (3.1) in $\mathcal{C}^2_\Omega$.

Lemma 4.6. *Let $0 < u \in H^1_{loc}(\mathcal{C}^1_\Omega)$ be a super-solution to* (3.1) *in $\mathcal{C}^1_\Omega$. Then*

$$u \ge c v_\psi \quad in \quad \mathcal{C}^2_{\Omega'}.$$

Proof. Choose a function $\theta(r) \in C^{0,1}[2,+\infty)$ such that $0 \le \theta(r) \le 1$, $\theta(2) = 1$ and $\theta(r) = 0$ for $r \ge 3$. Set $\tilde v_k := v_k - \theta\phi_{V,k}$. By direct computations,

$$\mathcal{E}_V(\tilde v_k, \tilde v_k) \le c_1|\alpha^-_{V,k}| + c_2 \qquad \text{and} \qquad \mathcal{E}_V(\tilde v_k, \tilde v_l) = 0 \quad \text{for} \;\; l \ne k. \tag{4.9}$$

Then it is straightforward that $\tilde v_k \in D^1_0(\mathcal{C}^2_\Omega)$. Consider the function

$$\tilde v_\psi := \sum_{k=1}^\infty \psi_k \tilde v_k.$$

By (4.9) and taking into account that $|\alpha^-_{V,k}| \asymp \sqrt{\lambda_k}$ we obtain

$$\begin{aligned}\mathcal{E}_V(\tilde v_\psi, \tilde v_\psi) &\leq \sum_{k=1}^\infty \psi_k^2(c_1|\alpha^-_{V,k}| + c_2)\\ &\leq c_3\Big(\sum_{k=1}^\infty \psi_k^2\lambda_k\Big)^{1/2}\Big(\sum_{k=1}^\infty \psi_k^2\Big)^{1/2} + c_2\sum_{k=1}^\infty \psi_k^2\\ &= c_3\|\nabla_\omega\psi\|_{L^2}\|\psi\|_{L^2} + c_2\|\psi\|^2_{L^2}.\end{aligned}$$

Hence, $\tilde v_\psi = v_\psi - \theta\psi \in D^1_0(\mathcal{C}^2_\Omega)$.

Now observe that by the weak Harnack inequality (4.5) there exists $\delta > 0$ such that $u > \delta$ in $\mathcal{C}^{(2,3)}_{\Omega'}$. Fix $c > 0$ such that $c\psi < \delta$ in Ω'. Thus $u > c\theta\psi$ in $\mathcal{C}^2_\Omega$. Represent

$$u - cv_\psi = (u - c\psi\theta) - c\tilde v_\psi,$$

where $\tilde v_\psi \in D^1_0(\mathcal{C}^2_\Omega)$ and notice that $u - v_\psi$ is a super-solution to (3.1). By Lemma A.8 we conclude that $u - c\psi\theta \geq c\tilde v_\psi$, that is, $u \geq cv_\psi$ in $\mathcal{C}^2_\Omega$. □

Lemma 4.7. *$m_{v_\psi}(R, \Omega') \asymp R^{\alpha^-_1}$ as $R \to \infty$.*

Proof. Choosing $u = r^{\alpha^-_{V,1}}\phi_1$ as a (super-) solution in Lemma 4.6 we immediately conclude that

$$m_{v_\psi}(R, \Omega') \leq cR^{\alpha^-_{V,1}} \quad \text{for } R \gg 2.$$

To obtain the reverse inequality, note that v_ψ as $v_\psi = \psi_1 v_1 + w_\psi$. Then by Lemma 4.1 we obtain the uniform bound

$$|w_\psi(r,\omega)| \leq \eta_2(r)\sum_{k=2}^\infty |\psi_k||\phi_k(\omega)| \leq cr^{\alpha^-_{V,2}}. \tag{4.10}$$

Note that $\phi_{V,1} > \delta$ in Ω', for some $\delta > 0$. We conclude that

$$m_{v_\psi}(\Omega', R) \geq c_2R^{\alpha^-_{V,1}} - c_3R^{\alpha^-_{V,2}} \quad \text{for } R \gg 4.$$

This completes the proof, since $\alpha^-_{V,2} < \alpha^-_{V,1} < 0$. □

Combining Lemmas 4.6 and 4.7 we obtain the lower bound in (4.3).

Upper estimate. Fix a subdomain $\Omega' \subseteq \Omega$ and a function $0 \le \psi \in C_c^\infty(\Omega')$. Let $R \ge 4$. For $(r,\omega) \in \mathcal{C}_\Omega^{(1,R)}$ and $k \in \mathbb{N}$ define

$$v_{k,R}(r,\omega) := \eta_{k,R}(r)\phi_{V,k}(\omega), \quad \text{where} \quad \eta_{k,R}(r) := \frac{r^{\alpha^+_{V,k}} - r^{\alpha^-_{V,k}}}{R^{\alpha^+_{V,k}} - R^{\alpha^-_{V,k}}}. \qquad (4.11)$$

Let $\theta : [0,1] \to \mathbb{R}$ be a smooth function such that $0 \le \theta \le 1$, $\theta(1) = 1$ and $\theta(\xi) = 0$ for $\xi \in [0,1/2]$. For $r \in [R/2,R]$ set $\theta_R(r) := \theta(r/R)$. A direct computation verifies that $v_{k,R}$ is a solution to the problem

$$\Big(-\Delta - \frac{V(\omega)}{|x|^2}\Big) v = 0, \qquad v - \theta_R\phi_{V,k} \in H_0^1(\mathcal{C}_\Omega^{(1,R)}). \qquad (4.12)$$

Let

$$v_{\psi,R} := \sum_{k=1}^\infty \psi_k \, v_{k,R},$$

where ψ_k are the Fourier coefficients of ψ as in (4.1). Thus, $v_{\psi,R} \in H^1_{loc}(\mathcal{C}_\Omega^{1,R})$ is a solution to (4.12) and $v_{\psi,R}(R,\omega) = \psi(\omega)$.

Fix a compact $K_0 \subset \mathcal{C}_\Omega^{(2,3)}$. Define the *comparison function* $v_{\psi,R}$ by

$$\tilde v_{\psi,R} := \frac{v_{\psi,R}}{\inf_{K_0} v_{\psi,R}}.$$

Then $\inf_{K_0} \tilde v_{\psi,R} = 1$. Note that the construction of $\tilde v_{\psi,R}$ depends only on the choice of K_0, ψ and R. The following lemma is a weak version of the Phragmén-Lindelöf comparison principle adopted to our framework.

Lemma 4.8. *Let $0 < u \in H^1_{loc}(\mathcal{C}^1_\Omega)$ be a super-solution to* (3.1) *in $\mathcal{C}^1_\Omega$. Then*

$$m_u(R,\Omega') \le cM_{\tilde v_{\psi,R}}(R,\Omega'), \qquad R \ge 4.$$

Proof. Set $\delta_R := \inf_{K_0} v_{\psi,R}$. For a contradiction assume that for any $c > 0$ there exists $R \ge 4$ such that

$$u \ge c\tilde v_{\psi,R} = \frac{c}{\delta_R} v_{\psi,R} \quad \text{in } \mathcal{C}_{\Omega'}^{(R/2,R)}.$$

Let $\psi_R > 0$ be the unique solution to the problem

$$-\Delta v - \frac{V(\omega)}{|x|^2} v = 0, \qquad v - \theta_R\psi \in H_0^1(\mathcal{C}_{\Omega'}^{(R/2,R)}).$$

Then clearly

$$\Big(-\Delta - \frac{V(\omega)}{|x|^2}\Big)(v_{\psi,R} - \psi_R) = 0 \quad \text{in } \mathcal{C}_{\Omega'}^{(R/2,R)}.$$

Observe that $v_{\psi,R} > 0$ in $\mathcal{C}_\Omega^{(1,R)} \setminus \mathcal{C}_{\Omega'}^{(R/2,R)}$. Hence $(v_{\psi,R} - \psi_R)^- \in D_0^1(\mathcal{C}_{\Omega'}^{R/2,R})$. By Lemma A.4 we conclude that

$$v_{\psi,R} \geq \psi_R \quad \text{in } \mathcal{C}_{\Omega'}^{(R/2,R)}.$$

Let $\bar{\psi}_R$ denote the function ψ_R, extended to $\mathcal{C}_\Omega^{(1,R)}$ by zero. Therefore

$$\begin{aligned}&\Big(-\Delta - \frac{V(\omega)}{|x|^2}\Big)(u - c\tilde{v}_{\psi,R})\\ &\quad= \Big(-\Delta - \frac{V(\omega)}{|x|^2}\Big)\Big((u - \frac{c}{\delta_R}\bar{\psi}_R) - \frac{c}{\delta_R}(v_{\psi,R} - \bar{\psi}_R)\Big) \geq 0 \quad \text{in } \mathcal{C}_\Omega^{(1,R)}.\end{aligned}$$

Then Lemma A.8 implies that

$$u \geq c\tilde{v}_{\psi,R} \quad \text{in } \mathcal{C}_\Omega^{(1,R)}.$$

Since $c > 0$ is arbitrary, we conclude that $\inf_{K_0} u = +\infty$. Hence, by the weak Harnack inequality (4.5), $u \equiv +\infty$ in $\mathcal{C}_\Omega^1$, which is a contradiction. $\square$

Lemma 4.9. *$M_{\tilde{v}_{\psi,R}}(R, \Omega') \asymp R^{\alpha_{V,1}^+}$ as $R \to \infty$.*

Proof. Choosing $u := r^{\alpha_{V,1}^+}\phi_1$ as a (super-) solution in Lemma 4.8 we conclude that

$$M_{\tilde{v}_{\psi,R}}(R, \Omega') \geq cR^{\alpha_{V,1}^+}, \qquad R \gg 1.$$

Now we estimate $M_{\tilde{v}_{\psi,R}}(R, \Omega')$ from above.

First, observe that Lemma A.8, Lemma 4.1 and the arguments, similar to those in Lemma 4.6 imply the upper bound

$$v_{\psi,R}(r, \omega) \leq c\eta_{1,R}(r)\phi_{V,1}(\omega) \quad \text{in } \mathcal{C}_\Omega^{(1,R)},$$

where $c > 0$ is chosen so that $\psi \leq c\phi_{V,1}$ in Ω. Clearly, if $\alpha_{V,1}^+ \geq 0$, then $\eta_{1,R}(r) \leq 1$. However, if $\alpha_{V,1}^+ < 0$, then $\eta_{1,R}(r)$ attains its maximum at $r_* \in (1, R)$ with $\eta_{1,R}(r_*) \to \infty$ as $R \to \infty$. Nevertheless, one can readily verify that

$$\max_{r \in [R/2,R]} \eta_{k,R}(r) \leq \max\{1, 2^{-\alpha_{V,1}^+}\}, \qquad R \gg 1.$$

Therefore,

$$M_{v_{\psi,R}}(\Omega, R) \leq c_1, \qquad R \gg 1.$$

To estimate $\inf_{K_0} v_{\psi,R}$ from below, note that

$$v_{\psi,R} = \psi_1 v_{1,R} + w_{\psi,R}\,, \qquad \text{where} \quad w_{\psi,R} = \sum_{k=2}^{\infty} \psi_k v_{k,R}\,.$$

Then by Lemma 4.1 similarly to (4.10) we obtain

$$\sup_{K_0}|w_{\psi,R}| \le \max_{r\in(2,3)} \eta_k(r)\sum_{k=2}^{\infty}|\psi_k||\phi_{V,k}(\omega)| \le \frac{c_1}{R^{\alpha^+_{V,2}} - R^{\alpha^-_{V,2}}}.$$

We conclude that

$$\inf_{K_0} v_{\psi,R} \ge \inf_{K_0} \psi_1 v_{1,R}(r) - \sup_{K_0}|w_{\psi,R}| \ge \frac{c_2}{R^{\alpha_1^+}} - \frac{c_3}{R^{\alpha_2^+}}.$$

This completes the proof since $\alpha^+_{V,2} > \alpha^+_{V,1}$. □

Combining Lemmas 4.8 and 4.9 we obtain the upper bound in (4.3).

4.2. **Case** $C_H + \lambda_{V,1} = 0$**.** Let $\rho \ge 1$. Then Hardy's inequality (3.2) implies that the form $\mathcal{E}_V$ satisfies the λ–property (A.5) with $\lambda(x) = \frac{1/4}{|x|^2\log^2|x|}$. Hence the extended Dirichlet space $\mathcal{D}(\mathcal{E}_V, \mathcal{C}^\rho_\Omega)$ is well defined (see Appendix A), and in particular, the comparison principle (Lemma A.8) is valid. We denote

$$\tilde D^1_0(\mathcal{C}^\rho_\Omega) := \mathcal{D}(\mathcal{E}_V, \mathcal{C}^\rho_\Omega).$$

The space $\tilde D^1_0(\mathcal{C}^2_\Omega)$ is larger than $D^1_0(\mathcal{C}^2_\Omega)$ (cf. [18, 42]). In order to see this, for $\beta \in [0,1]$ consider

$$v_\beta(r,\omega) := r^{\alpha_*}\log^\beta(r)\phi_{V,1}(\omega). \tag{4.13}$$

Clearly, $v_\beta \in C^\infty_{loc}(\mathcal{C}^\rho_\Omega)$ but $\nabla v_\beta \notin L^2(\mathcal{C}^\rho_\Omega)$. Let $\theta(r) \in C^{0,1}[\rho,+\infty)$ be such that $0 \le \theta(r) \le 1$, $\theta(\rho) = 1$ and $\theta(r) = 0$ for $r \ge \rho+1$.

Lemma 4.10. *$v_\beta - \theta\phi_{V,1} \in \tilde D^1_0(\mathcal{C}^\rho_\Omega)$ for each $\beta \in [0,1/2)$.*

Proof. Define the cut–off function $\theta_R(r) \in C^{0,1}_c(\mathcal{C}^1_\Omega)$ by

$$\theta_R(r) := \begin{cases} 1, & 1 \le r \le R, \\ \frac{\log(R^2/r)}{\log R}, & R \le r \le R^2, \\ 0, & r \ge R^2. \end{cases}$$

Let $w_R := \theta_R(v_\beta - \theta\phi_{V,1})$. According to Lemma A.9, one can represent $\mathcal{E}_V(v_R)$ as

$$\begin{aligned} \mathcal{E}_V(w_R,w_R) &= \int_\rho^\infty\int_\Omega\left|\nabla\Big(\frac{w_R}{v_0}\Big)\right|^2 v_0^2\,d\omega\,r^{N-1}dr \\ &= \int_\rho^\infty\left|\nabla\Big(\log^\beta(r)\theta_R(r)\Big)\right|^2 r\,dr \le c_1 + c_2\log^{2\beta-1}(R) \le c. \end{aligned}$$

Hence, $\mathcal{E}_V(w_{R_n}, w_{R_n})$ is a Cauchy sequence, for an appropriate choice of $R_n \to \infty$. Since $(w_{R_n}) \subset C^{0,1}_{loc}(\mathcal{C}^\rho_\Omega)$ converges pointwise to the function v_β, the assertion follows. $\square$

Now we are in a position to prove estimate (4.4).

Lower estimate. As before, fix a proper smooth subdomain $\Omega' \Subset \Omega$ and a function $0 \lneq \psi \in C^\infty_c(\Omega')$. For $(r,\omega) \in \mathcal{C}^2_\Omega$ and $k \in \mathbb{N}$ set

$$v_*(r,\omega) := c_* r^{\alpha_*} \phi_{V,1}(\omega),$$

where $c_* > 0$ chosen so that $v_*(2,\omega) = \phi_{V,1}(\omega)$. Clearly $v_* \in H^1_{loc}(\mathcal{C}^2_\Omega)$ is a solution to (3.1) in $\mathcal{C}^2_\Omega$.

Define the *comparison function* v_ψ by

$$v_\psi := \psi_1 v_* + \sum_{k=2}^{\infty} \psi_k v_k, \tag{4.14}$$

where ψ_k are the Fourier coefficients of ψ as in (4.1) and v_k with $k \geq 2$ are defined by (4.7). Thus $v_\psi(2,\omega) = \psi(\omega)$. Observe that for $k \geq 2$ the functions v_k are solutions to (3.1) in $\mathcal{C}^2_\Omega$. Hence $v_\psi \in H^1_{loc}(\mathcal{C}^2_\Omega)$ is a solution to (3.1) in $\mathcal{C}^2_\Omega$.

Lemma 4.11. *Let $0 < u \in H^1_{loc}(\mathcal{C}^1_\Omega)$ be a super-solution to (3.1) in $\mathcal{C}^1_\Omega$. Then*

$$u \geq c v_\psi \quad in \quad \mathcal{C}^2_{\Omega'}.$$

Proof. Similar to the proof of Lemma 4.6. $\square$

Lemma 4.12. $m_{v_\psi}(R,\Omega') \asymp R^{\alpha_*}$ *as $R \to \infty$.*

Proof. Similar to the proof of Lemma 4.7. $\square$

Combining Lemmas 4.11 and 4.12 we obtain the lower bound in (4.4).

Upper estimate. Fix a subdomain $\Omega' \subseteq \Omega$ and a function $0 \leq \psi \in C^\infty_c(\Omega')$. Let $R \geq 4$. For $(r,\omega) \in \mathcal{C}^{(1,R)}_\Omega$ and $k \in \mathbb{N}$ define

$$v_{*,R}(r,\omega) := \eta_{*,R}(r)\phi_{V,1}(\omega), \quad \text{where} \quad \eta_{*,R}(r) := \frac{\log(r)}{\log(R)}\left(\frac{r}{R}\right)^{\alpha_*}. \tag{4.15}$$

Let $\theta : [0,1] \to \mathbb{R}$ be a smooth function such that $0 \leq \theta \leq 1$, $\theta(1) = 1$ and $\theta(\xi) = 0$ for $\xi \in [0,1/2]$. For $r \in [R/2,R]$ set $\theta_R(r) := \theta(r/R)$. A direct computation verifies that $v_{*,R}$ and $v_{k,R}$ ($k \geq 2$), defined by (4.7) are solutions to the problems

$$\Big(-\Delta - \frac{V(\omega)}{|x|^2}\Big)v = 0, \qquad v - \theta_R \phi_{V,k} \in H_0^1(\mathcal{C}_\Omega^{(1,R)}). \tag{4.16}$$

Let

$$v_{\psi,R} := \psi_1\, v_{*,R} + \sum_{k=2}^{\infty} \psi_k\, v_{k,R}$$

where ψ_k are the Fourier coefficients of ψ as in (4.1). Thus $v_{\psi,R} \in H^1_{loc}(\mathcal{C}_\Omega^{1,R})$ is a solution to (4.16) and $v_\psi(R,\omega) = \psi(\omega)$.

Fix a compact $K_0 \subset \mathcal{C}_\Omega^{(2,3)}$. Define the *comparison function* $\tilde{v}_{\psi,R}$ by

$$\tilde{v}_{\psi,R} := \frac{v_{\psi,R}}{\inf_{K_0} v_{\psi,R}}.$$

Lemma 4.13. *Let* $0 < u \in H^1_{loc}(\mathcal{C}^1_\Omega)$ *be a super-solution to* (3.1) *in* $\mathcal{C}^1_\Omega$. *Then*

$$m_u(R,\Omega') < cM_{\tilde{v}_{\psi,R}}(R,\Omega'), \qquad R \geq 4.$$

Proof. Similar to the proof of Lemma 4.8. □

Lemma 4.14. $M_{\tilde{v}_{\psi,R}}(R,\Omega') \asymp R^{\alpha_*}\log(R)$ *as* $R \to \infty$.

Proof. Similar to the proof of Lemma 4.9. □

Combining Lemmas 4.13 and 4.14 we obtain the upper bound in (4.4).

4.3. **Auxiliary linear equation.** In this subsection we consider the inhomogeneous linear equation

$$-\Delta w - \frac{C_H + \lambda_1}{|x|^2} w = \frac{\psi(\omega)}{|x|^{2-\alpha_*}\log^\sigma |x|} \quad \text{in} \quad \mathcal{C}^\rho_\Omega, \tag{4.17}$$

where $\alpha_* = \frac{2-N}{2}$, $\sigma > 0$, $\rho > \exp(1)$ and $0 \lneq \psi \in C_c^\infty(\Omega)$.

Lemma 4.15. *Equation* (4.17) *has no positive super-solution for* $\sigma \leq 1$.

Proof. Without loss of generality we may assume $\sigma = 1$. For each $R \gg \rho$ we are going to construct a barrier $w_{\psi,R} > 0$ that solves the problem

$$-\Delta w - \frac{C_H + \lambda_1}{|x|^2} w = \frac{\psi(\omega)}{|x|^{2-\alpha_*}\log^\sigma |x|}, \qquad w \in H_0^1(\mathcal{C}_\Omega^{(\rho,R)}), \tag{4.18}$$

and blows up on a fixed compact $K \subset \mathcal{C}^\rho_\Omega$ as $R \to \infty$. Then by Lemma A.8,

$$u \geq w_{\psi,R} \qquad \text{in } \mathcal{C}_\Omega^{(\rho,R)}.$$

Therefore, we conclude that $u \equiv +\infty$ in K, which is a contradiction.

To construct such $w_{\psi,R}$, consider the boundary-value problem

$$-\eta_k'' - \frac{N-1}{r}\eta_k' - \frac{C_H - \delta_k^2}{r^2}\eta_k = \frac{1}{r^{2-\alpha_*}\log^\sigma(r)}, \quad \eta(\rho) = \eta(R) = 0, \quad (4.19)$$

where $\delta_k := \sqrt{\lambda_k - \lambda_1}$, and $k \in \mathbb{N}$. For $k = 1$, the solution to (4.19) is given by

$$\eta_{1,R}(r) = r^{\alpha_*}\left(A_{1,R} + B_{1,R}\log(r) + \log(r)\log\log(r)\right),$$

where

$$A_{1,R} = \frac{\log(R)\log(\rho)(\log\log(\rho) - \log\log(R))}{\log(R) - \log(\rho)},$$

$$B_{1,R} = \frac{\log(R)\log\log(R) - \log(\rho)\log\log(\rho)}{\log(R) - \log(\rho)}.$$

For every fixed $r_0 > \rho$, one sees that

$$\eta_{1,R}(r_0) \sim \log\log(R) \quad \text{as} \quad R \to \infty. \quad (4.20)$$

For $k \geq 2$ the solutions to (4.19) can be represented as

$$\eta_{k,R}(r) = A_{k,R}r^{\alpha_k^-} + B_{k,R}r^{\alpha_k^+} + \eta_k(r),$$

where

$$A_{k,R} = -\frac{R^{\alpha_k^+}\eta_k(\rho) - \rho^{\alpha_k^+}\eta_k(R)}{R^{\alpha_k^+}\rho^{\alpha_k^-} - R^{\alpha_k^-}\rho^{\alpha_k^+}}, \quad B_{k,R} = -\frac{R^{\alpha_k^-}\eta_k(\rho) - \rho^{\alpha_k^-}\eta_k(R)}{R^{\alpha_k^-}\rho^{\alpha_k^+} - R^{\alpha_k^+}\rho^{\alpha_k^-}},$$

and

$$\eta_k(r) := \frac{r^{\alpha_*}}{2\delta_k}\left(r^{\delta_k}\int_r^\infty \frac{t^{-\delta_k-1}}{\log^\sigma(t)}\,dt + r^{-\delta_k}\int_\rho^r \frac{t^{\delta_k-1}}{\log^\sigma(t)}\,dt\right). \quad (4.21)$$

It is easy to see that

$$0 < \eta_k \leq \frac{r^{\alpha_*}}{\delta_k^2\log^\sigma(r)}, \qquad \forall k \geq 2. \quad (4.22)$$

Moreover, $\eta_k(r) = O(r^{\alpha_*}\log^{-\sigma}(r))$ as $r \to \infty$.

Represent $\psi = \sum_{k=1}^\infty \psi_k\phi_k$ as in (4.1), and set

$$w_{\psi,R} = \sum_{k=1}^\infty \eta_{k,R}\psi_k\phi_k. \quad (4.23)$$

It is easy to see that the series converges in $H_0^1(\mathcal{C}_\Omega^{(\rho,R)})$ and $w_{\psi,R}$ solves (4.18). Fix a compact $K \subset \mathcal{C}_\Omega^\rho$. By Lemma 4.1 and in view of (4.22), we conclude that

$$\sup_K \Big|\sum_{k=2}^\infty \eta_{k,R}\psi_k\phi_k\Big| \le c\sup_K \frac{r^{\alpha_*}}{\delta_k^2\log^\sigma(r)} \le c_1, \qquad \forall R \gg \rho,$$

with constants $c, c_1 > 0$ that do not depend on R. Therefore,

$$\inf_K w_{\psi,R} \sim \inf_K \{\eta_{1,R}\psi_1\phi_1\} \sim \log\log(R) \to \infty \quad \text{as} \quad R \to \infty,$$

by (4.20), and the assertion follows. □

Remark 4.16. The value of $\sigma = 1$ in the above lemma is sharp. When $\sigma > 1$ it is not difficult to construct solutions to (4.17) in the form (4.23) with

$$\eta_1(r) := \frac{r^{\alpha_*}\log^{2-\sigma}(r)}{-(\sigma^2 - 3\sigma + 2)}$$

and $\eta_k(r)$ as in (4.21). Alternatively, if $\sigma > 3/2$, then Hardy's inequality (3.2) implies that the quadratic form corresponding to (4.17) satisfies the λ–property with $\lambda(x) = \frac{1/4}{|x|^2\log^2|x|}$. Further, $|x|^{\alpha_*-2}\log^{-\sigma}|x| \in L^2(\lambda^{-1}dx)$ for any $\sigma > 3/2$. Thus Lemmas A.3 and A.4 imply that (4.17) has a positive solution in $D_0^1(\mathcal{C}_\Omega^\rho)$.

5. Proof of Theorem 2.2, superlinear case $p \ge 1$

In the proof we distinguish between the two cases: $\mu < C_H + \lambda_1$ and $\mu = C_H + \lambda_1$.

5.1. Case $\mu < C_H + \lambda_1$.

Nonexistence. First we prove the nonexistence of super-solutions in the subcritical case, i.e. for (p, s) below the critical line Λ.

Lemma 5.1. *Let $p \ge 1$ and $s < \alpha^-(p-1) + 2$. Then* (2.1) *has no positive super-solutions in $\mathcal{C}_\Omega^1$.*

Proof. Let $w > 0$ be a super-solution to (2.1) in $\mathcal{C}_\Omega^1$. Then w is a super-solution to the linear equation

$$-\Delta w - \frac{\mu}{|x|^2}w = 0 \quad \text{in} \quad \mathcal{C}_\Omega^1. \tag{5.1}$$

Choose a proper subdomain $\Omega' \Subset \Omega$. Then, by Theorem 4.2, there exists $c > 0$ such that

$$m_w(R, \Omega') \ge cR^{\alpha^-} \quad (R \ge 2).$$

Linearizing (2.1) and using the bound above, we conclude that $w > 0$ is a super-solution to

$$-\Delta w - \frac{\mu}{|x|^2} w - \frac{V(x)}{|x|^2} w = 0 \quad \text{in} \quad \mathcal{C}^2_{\Omega'}, \tag{5.2}$$

where $V(x) := Cw^{p-1}|x|^{2-s}$ satisfies

$$V(x) \geq c^{p-1}|x|^{\alpha^-(p-1)+(2-s)} \quad \text{in} \quad \mathcal{C}^2_{\Omega'}.$$

Then the assertion follows by Corollary 3.2. □

Now we consider the critical case, i.e., when (p, s) belongs to the critical line Λ.

Lemma 5.2. *Let $p \geq 1$ and $s = \alpha^-(p-1) + 2$. Then* (2.1) *has no positive super-solutions in $\mathcal{C}^1_\Omega$.*

Proof. Let $w > 0$ be a super-solution to (2.1) in $\mathcal{C}^1_\Omega$. Choose a proper subdomain $\Omega' \Subset \Omega$. Arguing as in the proof above we conclude that w is a super-solution to (5.2) with $V(x) := Cw^{p-1}|x|^{2-s} \geq \delta$ in $\mathcal{C}^2_{\Omega'}$, for some $\delta > 0$. Thus w is a super-solution to the linear equation

$$-\Delta w - \frac{W(\omega)}{|x|^2} w = 0 \quad \text{in} \quad \mathcal{C}^2_{\Omega}, \tag{5.3}$$

where $W(\omega) := \mu + \varepsilon\chi_{\Omega'}$, with a fixed $\varepsilon \in (0, \delta]$. By the variational characterization of the principal Dirichlet eigenvalue of $-\Delta_\omega - \mu - \varepsilon\chi_{\Omega'}$ on Ω and since $\mu < C_H + \lambda_1$, one can choose a small $\varepsilon > 0$ so that $C_H + \lambda_{W,1} > 0$. Applying Theorem 4.2 to (5.3) we conclude that

$$m_w(R, \Omega') \geq cr^{\alpha^-_{W,1}}, \quad R \geq 4,$$

with $\alpha^-_1 < \alpha^-_{W,1} < \alpha_*$. Therefore $w > 0$ is a super-solution to

$$-\Delta w - \frac{\mu}{|x|^2} w - \frac{\tilde{V}(x)}{|x|^2} w = 0 \quad \text{in} \quad \mathcal{C}^4_{\Omega'},$$

where $\tilde{V}(x) := Cw^{p-1}|x|^{2-s}$. Therefore,

$$\tilde{V}(x) \geq Cc^{p-1}|x|^{\alpha^-_{W,1}(p-1)+(2-s)} \quad \text{in} \quad C^4_{\Omega'},$$

with $\alpha^-_{W,1}(p-1) + (2-s) > 0$. Then the assertion follows from Corollary 3.2. □

Existence. Let $p > 1$ and $s > \alpha_1^-(p-1)+2$. Choose $\alpha \in (\alpha_1^-, \alpha_1^+)$ such that $\alpha \le \frac{s-2}{p-1}$. Then one can verify directly that the functions

$$w := \tau r^{\alpha}\phi_1(\omega)$$

are super-solutions to (2.1) in $\mathcal{C}^1_\Omega$ for sufficiently small $\tau > 0$. In the case $p = 1$ and $s > 2$ one sees that for any $\alpha \in (\alpha_1^-, \alpha_1^+)$ the function w is a super-solution to (2.1) in $\mathcal{C}^\rho_\Omega$ with a sufficiently large $\rho \gg 1$.

5.2. **Case** $\mu = C_H + \lambda_1$**.**

Nonexistence. The proof can be performed in one step for both subcritical and critical cases.

Lemma 5.3. *Let $p \ge 1$ and $s \le \alpha_*(p-1)+2$. Then* (2.1) *has no positive super-solutions in $\mathcal{C}^1_\Omega$.*

Proof. Assume that $w > 0$ is a super-solution to (2.1) in $\mathcal{C}^1_\Omega$. Then w is a super-solution to

$$-\Delta w - \frac{C_H + \lambda_1}{|x|^2} w = 0 \quad \text{in} \quad \mathcal{C}^1_\Omega.$$

Choose a proper subdomain $\Omega' \Subset \Omega$. Then by Theorem 4.2

$$m_w(R, \Omega') \ge cR^{\alpha_*}, \quad R \ge 2.$$

Linearizing (2.1) and using the bound above, we conclude that $w > 0$ is a super-solution to

$$-\Delta w - \frac{C_H + \lambda_1}{|x|^2} w - \frac{W(x)}{|x|^2} w = 0 \quad \text{in} \quad \mathcal{C}^2_{\Omega'}, \tag{5.4}$$

where $W(x) := Cw^{p-1}|x|^{2-s} \ge \tilde{c}$ in $C^2_{\Omega'}$. Then the assertion follows from Corollary 3.2. $\square$

Existence. Let $p > 1$ and $s > \alpha_*(p-1)+2$. Choose $\beta \in (0,1)$. Then one verifies directly that the functions

$$w := \tau r^{\alpha_*} \log^{\beta}(r)\phi_1(\omega)$$

are super-solutions to (2.1) in $\mathcal{C}^1_\Omega$ for sufficiently small $\tau > 0$. In the case $p = 1$ and $s > 2$ one has to choose $\rho \gg 1$ sufficiently large.

6. Proof of Theorem 2.2, sublinear case $p < 1$

As before, we consider separately the cases $\mu < C_H + \lambda_1$ and $\mu = C_H + \lambda_1$. First, we sketch the proofs of two auxiliary lemmas.

Lemma 6.1. *Let $p < 1$. Let $w > 0$ be a super-solution to (2.1) in $\mathcal{C}^1_\Omega$. Then for each proper subdomain $\Omega' \Subset \Omega$ there exists $c > 0$ such that*

$$m_w(R, \Omega') \geq c\, R^{\frac{2-s}{1-p}}, \qquad R \gg 1. \tag{6.1}$$

Proof. Let $w > 0$ be a super-solution to (2.1). Then $-\Delta w \geq 0$ in $\mathcal{C}^1_\Omega$ and, by the weak Harnack inequality (see, e.g. [21, Theorem 8.18]), for any $s > 0$ and for any compact $K \subset \mathcal{C}^1_\Omega$ there exists $c > 0$ such that

$$\sup_K w^{-1} \leq c \left(R^{-N} \int_K w^{-s} dx \right)^{1/s}. \tag{6.2}$$

Further, it follows from Lemma A.9 that

$$\int_{\mathcal{C}^1_\Omega} |\nabla \varphi|^2 dx - \mu \int_{\mathcal{C}^1_\Omega} \frac{\varphi^2}{|x|^2} dx \geq C \int_{\mathcal{C}^1_\Omega} \frac{w^{p-1}}{|x|^s}\, \varphi^2 dx, \tag{6.3}$$

for all $\varphi \in H^1_c(\mathcal{C}^1_\Omega) \cap H^\infty_c(\mathcal{C}^1_\Omega)$. Fix a proper subdomain $\Omega' \Subset \Omega$. Choose $\psi \in C^\infty_c(\Omega)$ such that $\psi = 1$ on Ω'. Choose $\theta_R(r) \in C^{0,1}_c(1, +\infty)$ such that $0 \leq \theta_R \leq 1$, $\theta_R = 1$ for $r \in [R/2, R]$, $supp(\theta_R) = [R/4, 2R]$ and $|\nabla \theta_R| < c/R$. Then

$$\int_{\mathcal{C}^1_\Omega} |\nabla (\theta_R \psi)|^2 dx - \mu \int_{\mathcal{C}^1_\Omega} \frac{|\theta_R \psi|^2}{|x|^2} dx \leq c R^{N-2}. \tag{6.4}$$

On the other hand,

$$\int_{\mathcal{C}^1_{\Omega'}} \frac{w^{p-1}}{|x|^s} (\theta_R \psi)^2\, dx \geq \int_{\mathcal{C}^{(R/2,R)}_{\Omega'}} \frac{w^{p-1}}{|x|^s} dx \geq R^{-s} \int_{\mathcal{C}^{(R/2,R)}_{\Omega'}} w^{p-1} dx. \tag{6.5}$$

Combining (6.3), (6.4) and (6.5) we derive

$$c R^{s-2} \geq R^{-N} \int_{\mathcal{C}^{(R/2,R)}_{\Omega'}} w^{p-1} dx.$$

By (6.2) we obtain

$$c R^{\frac{s-2}{1-p}} \geq \left(R^{-N} \int_{\mathcal{C}^{(R/2,R)}_{\Omega'}} w^{-(1-p)}\, dx \right)^{\frac{1}{1-p}} \geq c_1 \sup_{\mathcal{C}^{(R/2,R)}_{\Omega'}} w^{-1}.$$

Hence the assertion follows. $\square$

Lemma 6.2. *Let $p < 1$, $\mu \le C_H + \lambda_1$ and $s \in \mathbb{R}$. Assume that* (2.1) *has a positive super-solution in $\mathcal{C}^1_\Omega$. Then there exists a positive solution to* (2.1) *in $\mathcal{C}^1_\Omega$.*

Proof. Let $w > 0$ be a super-solution to (2.1) in $\mathcal{C}^1_\Omega$. Then, by Lemma 4.6 or 4.11, $w \ge cv_\psi$ in $\mathcal{C}^1_\Omega$, where v_ψ is a comparison function defined by (4.8) or (4.14). Obviously, $v_\psi > 0$ is a sub-solution to (2.1) in $\mathcal{C}^2_\Omega$. Thus we can proceed via the standard sub and super-solutions techniques to prove existence of a solution to (2.1) in $\mathcal{C}^2_\Omega$, located between cv_ψ and w (cf. [26, Proposition 1.1(iii)]). Finally, after a suitable scaling we obtain a solution to (2.1) in $\mathcal{C}^1_\Omega$. □

6.1. **Case** $\mu < C_H + \lambda_1$.

Nonexistence. We shall distinguish between the subcritical and critical cases. When (p, s) is below the critical line, the proof of the nonexistence is straightforward.

Lemma 6.3. *Let $p < 1$ and $s < \alpha_1^+(p-1) + 2$. Then* (2.1) *has no positive super-solutions in $\mathcal{C}^1_\Omega$.*

Proof. Let $w > 0$ be a super-solution to (2.1). Then w is a super-solution to the linear equation

$$-\Delta w - \frac{\mu}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^1_\Omega\,. \tag{6.6}$$

Choose a proper subdomain $\Omega' \Subset \Omega$. By Theorem 4.2 we conclude that

$$m_w(R, \Omega') \le cR^{\alpha_1^+}, \qquad R \gg 1. \tag{6.7}$$

This contradicts to (6.1). □

Next we consider the case when (p, s) is on the critical line, and hence (6.7) is no longer incompatible with (6.1).

Lemma 6.4. *Let $p < 1$ and $s = \alpha_1^+(p-1) + 2$. Then* (2.1) *has no positive super-solutions in $\mathcal{C}^1_\Omega$.*

Proof. Let $w > 0$ be a super-solution to (2.1). According to Lemma 6.2 we may assume that w is a solution to (2.1). Choose a proper subdomain $\Omega' \Subset \Omega$. Linearizing (2.1) and using the upper bound (6.1) we conclude that $w > 0$ is a solution to

$$-\Delta w - \frac{\mu}{|x|^2} w - \frac{V(x)}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^1_\Omega, \tag{6.8}$$

where $V(x) := c^{p-1}|x|^{2-s}w^{p-1}$ satisfies $V(x) \le c_1$ in $\mathcal{C}^{\rho_1}_{\Omega'}$, with a fixed $\rho_1 \gg 1$. This implies, in particular, that w satisfies a strong Harnack's inequality with r–independent constants. More precisely, for a given subdomain $\Omega'' \Subset \Omega'$ one has

$$M_w(R,\Omega'') \le C_s\, m_w(R,\Omega''), \qquad R \gg \rho, \tag{6.9}$$

where $C_s = C_s(\Omega'') > 0$ does not depend on $R \gg \rho$. Using (6.9) and the upper bound (6.7) we conclude that

$$M_w(R,\Omega'') \le c_2 R^{\alpha_1^+}, \qquad R \gg \rho. \tag{6.10}$$

This implies that $V(x) \ge \delta$ in $\mathcal{C}^{\rho_2}_{\Omega''}$, for some $\delta > 0$ and $\rho_2 \gg \rho_1$. Hence $w > 0$ is a super-solution to the linear equation

$$-\Delta w - \frac{W_\varepsilon(\omega)}{|x|^2} w = 0 \quad \text{in } \mathcal{C}^{\rho_2}_{\Omega}, \tag{6.11}$$

where $W_\varepsilon(\omega) := \mu + \varepsilon\chi_{\Omega''}$, with a fixed $\varepsilon \in (0,\delta]$. By the variational characterization of the principal Dirichlet eigenvalue of $-\Delta_\omega - W_\varepsilon$ on Ω and since $\mu < C_H + \lambda_1$, one can choose a small $\varepsilon > 0$ such that $C_H + \lambda_{W_\varepsilon,1} > 0$. Applying Theorem 4.2 to (6.11) we conclude that

$$m_w(R,\Omega') \le c_2 R^{\alpha^+_{W_\varepsilon,1}}, \tag{6.12}$$

with $\alpha^+_{W_\varepsilon,1} < \alpha_1^+$. Now (6.12) contradicts the lower bound (6.1). $\square$

Existence. Let $s > \alpha_1^+(p-1)+2$. Assume that $0 \le p < 1$. Choose $\alpha \in (\alpha_1^-, \alpha_1^+)$ such that $\alpha \ge \frac{s-2}{p-1}$. Then there exists a unique bounded positive solution to the problem

$$-\Delta_\omega \phi - (\alpha(\alpha+N-2)+\mu)\phi = 1, \qquad \phi \in H^1_0(\Omega).$$

Further, a direct computation verifies that the functions $w := \tau r^\alpha \phi(\omega)$ are super-solutions to (2.1) in $\mathcal{C}^1_\Omega$ for a sufficiently large $\tau > 0$.

Now assume that $p < 0$. Choose α as above, so that there exists a unique bounded positive solution of the problem

$$-\Delta_\omega \bar\phi - (\alpha(\alpha+N-2)+\mu)(\bar\phi+1) = 1, \qquad \bar\phi \in H^1_0(\Omega).$$

Then $w := \tau r^\alpha \bar\phi(\omega)$ are super-solutions to (2.1) in $\mathcal{C}^1_\Omega$ for sufficiently large $\tau > 0$.

6.2. **Case $\mu = C_H + \lambda_1$.**

Nonexistence. The proof is straightforward when (p,s) is below the critical line Λ.

Lemma 6.5. *Let $p<1$ and $s<\alpha_*(p-1)+2$. Then* (2.1) *has no positive super-solutions in $\mathcal{C}^1_\Omega$.*

Proof. Let $w>0$ be a super-solution to (2.1). Similarly to the proof of Lemma 6.3, by Theorem 4.2 we conclude that for a proper subdomain $\Omega' \Subset \Omega$,

$$m_w(R,\Omega') \le cR^{\frac{2-N}{2}}\log(R), \qquad R \gg 1. \tag{6.13}$$

This contradicts the lower bound (6.1). $\square$

When (p,s) belongs to the critical line Λ inequality (6.13) is no longer incompatible with (6.1).

Lemma 6.6. *Let $p\in[-1,1)$ and $s=\alpha_*(p-1)+2$. Then* (2.1) *has no positive super-solutions in $\mathcal{C}^1_\Omega$.*

Proof. Let $w>0$ be a super-solution to (2.1). If $p\in[0,1)$, then the lower bound (6.1) implies that w is a super-solution to

$$-\Delta w-\frac{C_H+\lambda_1}{|x|^2}w=\frac{\psi(\omega)}{|x|^{2-\alpha_*}} \quad \text{in} \quad \mathcal{C}^{\rho'}_\Omega, \tag{6.14}$$

with some $\psi\in C^\infty_c(\Omega)$ and $\rho'>\rho$. From Lemma 4.15, it follows that (6.14) has no positive super-solutions.

Let $p\in[-1,0)$. According to Lemma 6.2 we may assume that w is a solution to (2.1). Similarly to the proof of Lemma 6.4, for a proper subdomain $\Omega'\Subset\Omega$ and a function $\psi\in C^\infty_c(\Omega')$ we conclude that $w>0$ is a super-solution to the linear equation

$$-\Delta w-\frac{C_H+\lambda_1}{|x|^2}w=\frac{\psi(\omega)}{|x|^{2-\alpha_*}\log^{-p}|x|} \quad \text{in} \quad \mathcal{C}^{\rho'}_\Omega, \tag{6.15}$$

for some $\rho'>\rho$. Then the assertion follows from Lemma 4.15. $\square$

Existence. In the critical case $\mu=C_H+\lambda_1$ positive super-solutions to (2.1) with $p<1$ can not be constructed as “pseudo”–radial functions of the form $u=v(r)\varphi(\omega)>0$, as the following proposition shows.

Proposition 6.7. *Let $u=v(r)\varphi(\omega)>0$ be a super-solution to*

$$-\Delta u-\frac{C_H+\lambda_1}{|x|^2}u=0 \quad in \quad \mathcal{C}^{\rho}_\Omega. \tag{6.16}$$

Then $u = v(r)\phi_1(\omega)$, where v is a super-solution to

$$-\frac{\partial^2 v}{\partial r^2} - \frac{N-1}{r}\frac{\partial v}{\partial r} - \frac{C_H}{r^2} v \geq 0 \quad \textit{in} \quad (\rho, \infty). \tag{6.17}$$

Proof. Let $u = v(r)\varphi(\omega) > 0$ be a super-solution to (6.16). Then

$$\left\{-\frac{\partial^2 v}{\partial r^2} - \frac{N-1}{r}\frac{\partial v}{\partial r} - \frac{C_H}{r^2} v\right\}\varphi + \{(-\Delta_\omega - \lambda_1)\varphi\}\frac{v}{r^2} \geq 0 \quad \text{in} \quad \mathcal{C}_\Omega^\rho.$$

Separating the variables and using Barta's inequality (see Lemma A.9)

$$\sup_{0<\varphi\in H^1_{loc}(\Omega)}\left\{\inf_{\omega\in\Omega}\frac{(-\Delta_\omega - \lambda_1)\varphi}{\varphi}\right\} \leq 0,$$

we obtain

$$\frac{r^2}{v}\left\{-\frac{\partial^2 v}{\partial r^2} - \frac{N-1}{r}\frac{\partial v}{\partial r} - \frac{C_H}{r^2} v\right\} \geq -\left\{\inf_{\omega\in\Omega}\frac{(-\Delta_\omega - \lambda_1)\varphi}{\varphi}\right\} \geq 0 \quad \text{in} \quad \mathcal{C}_\Omega^\rho.$$

On the other hand, for $\epsilon \geq 0$, the one–dimensional Hardy's inequality implies that the inequality

$$-\frac{\partial^2 v}{\partial r^2} - \frac{N-1}{r}\frac{\partial v}{\partial r} - \frac{C_H}{r^2} v \geq \epsilon\frac{v}{r^2} \quad \text{in} \quad (\rho, \infty)$$

has a positive solution if and only if $\epsilon = 0$. Hence

$$\inf_{\omega\in\Omega}\frac{(-\Delta_\omega - \lambda_1)\varphi}{\varphi} = 0,$$

and therefore, $\varphi = \phi_1$. We conclude that u must be of the form $u = v(r)\phi_1(\omega)$, where v is a super-solution to (6.17). □

It is easy to see that if $\Omega \Subset S^{N-1}$ is a proper subdomain of the sphere, then equation (2.1) with $p < 1$ does not admit positive super-solutions of the form $v(r)\phi_1(\omega)$. Nevertheless, for (p, s) above the critical line Λ we prove the following.

Lemma 6.8. *Let $p < 1$ and $s > \alpha_*(p-1) + 2$. Then there exists a positive super-solution to (2.1).*

Proof. Given $\varepsilon \in (0, 1/4)$, $\sigma > 3/2$ and $\rho \geq \exp(1)$, consider the problem

$$-\Delta w - \frac{C_H + \lambda_1}{|x|^2} w - \frac{\varepsilon}{|x|^2 \log^2|x|} w = \frac{1}{|x|^{2-\alpha_*}\log^\sigma|x|}, \quad w \in \mathcal{D}(\mathcal{E}_V, \mathcal{C}_\Omega^\rho). \tag{6.18}$$

It follows from Hardy's inequality (3.2) that the quadratic form that corresponds to (6.18) satisfies the λ–property with $\lambda(x) = \frac{1/4-\varepsilon}{|x|^2\log^2|x|}$. Further,

$|x|^{\alpha_*-2}\log^{-\sigma}|x| \in L^2(\lambda^{-1}dx)$. Thus Lemmas A.3 and A.4 imply that (6.18) has a unique solution $w_\sigma > 0$. Choose $\beta > \sigma$ and set

$$v_\sigma := w_\sigma + \frac{|x|^{\alpha_*}}{\log^\beta |x|}.$$

Then

$$\begin{aligned}\left(-\Delta - \frac{C_H+\lambda_1}{|x|^2} - \frac{\varepsilon}{|x|^2\log^2|x|}\right) v_\sigma \\ = \frac{1}{|x|^{2-\alpha_*}}\left(\frac{1}{\log^\sigma|x|} - \frac{\lambda_1}{\log^\beta|x|} - \frac{\beta(1+\beta)+\varepsilon}{\log^{\beta+2}|x|}\right) \geq 0 \text{ in } \mathcal{C}_\Omega^{\rho'},\end{aligned}$$

for some $\rho' \gg \rho$. Set $\delta := s - \alpha_*(p-1) - 2 > 0$ and choose $\tau = \tau(\delta) > 0$ such that

$$\frac{C}{|x|^s}(\tau v_\sigma)^{p-1} \leq \frac{C\tau^{p-1}\log^{\beta(1-p)+2}|x|}{\varepsilon|x|^\delta}\frac{\varepsilon}{|x|^2\log^2|x|} \leq \frac{\varepsilon}{|x|^2\log^2|x|} \quad \text{in} \quad \mathcal{C}_\Omega^{\rho'}.$$

Then

$$\begin{aligned}\left(-\Delta - \frac{C_H+\lambda_1}{|x|^2}\right)(\tau v_\sigma) &\geq \frac{\varepsilon}{|x|^2\log^2|x|}(\tau v_\sigma) \geq \frac{C}{|x|^s}(\tau v_\sigma)^{p-1}(\tau v_\sigma) \\ &= \frac{C}{|x|^s}(\tau v_\sigma)^p \quad \text{in} \quad \mathcal{C}_\Omega^{\rho'},\end{aligned}$$

that is, $\tau v_\sigma > 0$ is a super-solution to (2.1) in $\mathcal{C}_\Omega^{\rho'}$. □

In the case of exterior domains, the existence of positive super-solutions on the critical line Λ for $p < -1$ is elementarily observed.

Lemma 6.9. *Assume that* $\Omega = S^{N-1}$. *Let* $p < -1$ *and* $s = \alpha_*(p-1)+2$. *Choose* $\beta \in (\frac{2}{1-p}, 1)$. *Then*

$$v := \tau|x|^{\alpha_*}\log^\beta|x|$$

is a super-solution to (2.1) *in* $\mathcal{C}_\Omega^\rho$ *for sufficiently large* $\tau > 0$.

Proof. Follows from a direct computation. □

In the case of proper domains $\Omega \Subset S^{N-1}$, the existence (or nonexistence) of positive super-solutions to (2.1) with $p < -1$ and $s = \alpha_*(p-1)+2$ becomes a more delicate issue that remains open at the moment. The analysis of the decay rate of super-solutions to (6.16) near the lateral boundary of the cone should be invoked. We will return to this problem elsewhere.

APPENDIX A

Let $\mathcal{E}_V$ be a symmetric bilinear form defined by

$$\mathcal{E}_V(u,v) := \int_G \nabla u \cdot \nabla v\,dx - \int_G Vuv\,dx \qquad (u,v \in H^1_c(G) \cap L^\infty_c(G)),$$

where $G \subseteq \mathbb{R}^N$ is a domain and $V = V^+ - V^-$ is a potential, such that $0 \le V^+ \in L^1_{loc}(G)$ and $0 \le V^- \in L^\infty_{loc}(G)$. Below we present several facts concerning the relations between the positivity of the form $\mathcal{E}_V$ and the existence of positive (super-) solutions to the linear equation

$$(-\Delta - V)v = f \quad \text{in} \quad G, \tag{A.1}$$

associated with $\mathcal{E}_V$, where $f \in L^1_{loc}(G)$. A super-solution to (A.1) is a function $u \in H^1_{loc}(G) \cap L^1_{loc}(G, V\,dx)$ such that

$$\int_G \nabla u \cdot \nabla \varphi\,dx - \int_G Vu\varphi\,dx \ge \int_G f\varphi\,dx, \tag{A.2}$$

for all $0 \le \varphi \in H^1_c(G) \cap L^\infty_c(G)$. The notions of a sub-solution and solution are defined similarly by replacing "$\ge$" with "$\le$" and "$=$" respectively. Most of the results below are extracted from the Agmon's criticality theory (cf. [2, 3]) in combination with the facts from the theory of Dirichlet forms (cf. [15, 19]). We include the proofs for the completeness of the exposition.

Extended Dirichlet Space. Assume that the form $\mathcal{E}_V$ is *positive definite*; that is

$$\mathcal{E}_V(u,u) > 0, \qquad \forall\, 0 \ne u \in H^1_c(G) \cap L^\infty_c(G). \tag{A.3}$$

Following Fukushima [19, p.35–36], denote by $\mathcal{D}(\mathcal{E}_V, G)$ the family of measurable almost everywhere finite functions $u : G \to \mathbb{R}$ such that there exists an $\mathcal{E}_V$–Cauchy sequence $(u_n) \subset H^1_c(G) \cap L^\infty_c(G)$ that converges to u almost everywhere in G. This sequence (u_n) is called an *approximating sequence* for $u \in \mathcal{D}(\mathcal{E}_V, G)$. Then the limit $\mathcal{E}_V(u,u) := \lim_{n\to\infty} \mathcal{E}_V(u_n,u_n)$ exists and is independent of the choice of the approximating sequence. Thus $\mathcal{E}_V$ is extended uniquely to a nonnegative definite bilinear form on $\mathcal{D}(\mathcal{E}_V, G)$. The family $\mathcal{D}(\mathcal{E}_V, G)$ is called the *extended Dirichlet space* of $\mathcal{E}_V$. It is not a Hilbert space, in general. However, $\mathcal{D}(\mathcal{E}_V, G)$ is invariant under the standard truncations.

Lemma A.1. *Let $u \in \mathcal{D}(\mathcal{E}_V, G)$. Then $u^+ = u \vee 0 \in \mathcal{D}(\mathcal{E}_V, G)$, $u^- = -(u \wedge 0) \in \mathcal{D}(\mathcal{E}_V, G)$ and*

$$\mathcal{E}_V(u^\pm, u^\pm) \le \mathcal{E}(u,u), \qquad \forall u \in \mathcal{D}(\mathcal{E}_V, G). \tag{A.4}$$

If $u, v \in \mathcal{D}(\mathcal{E}_V, G)$, then $u \vee v, u \wedge v \in \mathcal{D}(\mathcal{E}_V, G)$.

Proof. Assume $u \in H^1_c(G) \cap L^\infty_c(G)$. Then $u^+ \in H^1_c(G) \cap L^\infty_c(G)$. By a direct computation we have

$$\mathcal{E}_V(u^+, u^+) + \mathcal{E}_V(u^-, u^-) = \mathcal{E}_V(u, u).$$

Hence (A.4) follows by (A.3) for any $u \in H^1_c(G) \cap L^\infty_c(G)$, and, then, for arbitrary $u \in \mathcal{D}(\mathcal{E}_V, G)$ by a standard approximation argument. $\square$

Remark A.2. *We do not claim that $u \in \mathcal{D}(\mathcal{E}_V, G)$ implies $u \wedge 1 \in \mathcal{D}(\mathcal{E}_V, G)$.*

Following [3, 4], we say that the form $\mathcal{E}_V$ satisfies the *λ–property* if there exists a function $0 < \lambda \in L^1_{loc}(G)$ such that $\lambda^{-1} \in L^\infty_{loc}(G)$ and

$$\mathcal{E}_V(u, u) \geq \int_G u^2 \lambda(x)\, dx, \qquad \forall\, u \in H^1_c(G) \cap L^\infty_c(G). \tag{A.5}$$

If $\mathcal{E}_V$ satisfies the λ–property, then the extended Dirichlet space $\mathcal{D}(\mathcal{E}_V, G)$ is a Hilbert space with the inner product $\mathcal{E}_V(\cdot, \cdot)$ and the corresponding norm $\| \cdot \|_{\mathcal{D}} = \sqrt{\mathcal{E}_V(\cdot, \cdot)}$. Clearly

$$H^1_c(G) \cap L^\infty_c(G) \subset \mathcal{D}(\mathcal{E}_V, G) \subset H^1_{loc}(G) \quad \text{and} \quad \mathcal{D}(\mathcal{E}_V, G) \subset L^2(G, \lambda\, dx).$$

By $\mathcal{D}'(\mathcal{E}_V, G)$ we denote the space of linear continuous functionals on the space $\mathcal{D}(\mathcal{E}_V, G)$. The following lemma is a standard consequence of the Riesz representation theorem.

Lemma A.3. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $l \in \mathcal{D}'(\mathcal{E}_V, G)$. Then there exists a unique $\phi_* \in \mathcal{D}(\mathcal{E}_V, G)$ such that*

$$\mathcal{E}_V(\phi_*, \varphi) = l(\varphi), \qquad \forall\, \varphi \in \mathcal{D}(\mathcal{E}_V, G). \tag{A.6}$$

Denote

$$\hat{\mathcal{D}}'(\mathcal{E}_V, G) := \{ f \in L^1_{loc}(G) : \int_G f\varphi\, dx \leq c\|\varphi\|_{\mathcal{D}}\,,\ \forall \varphi \in H^1_c(G) \cap L^\infty_c(G) \}.$$

It is easy to see that

$$L^2(G, \lambda^{-1}\, dx) \subset \hat{\mathcal{D}}'(\mathcal{E}_V, G).$$

Clearly $\hat{\mathcal{D}}'(\mathcal{E}_V, G)$ can be identified with a linear subspace of $\mathcal{D}'(\mathcal{E}_V, G)$. Thus Lemma A.3 implies that for any $f \in \hat{\mathcal{D}}'(\mathcal{E}_V, G)$ the problem

$$(-\Delta - V)u = f, \qquad u \in \mathcal{D}(\mathcal{E}_V, G), \tag{A.7}$$

has a unique solution.

Maximum and comparison principles. Consider the equation

$$(-\Delta - V)u = 0 \quad \text{in } G. \tag{A.8}$$

We present weak maximum and comparison principles for solutions and super-solutions of (A.1) in a form suitable for our framework.

Lemma A.4. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $w \in H^1_{loc}(G)$ be a super-solution to* (A.8) *such that $w^- \in \mathcal{D}(\mathcal{E}_V, G)$. Then $w \geq 0$ in G.*

Proof. Let $(\varphi_n) \subset H^1_c(G) \cap L^\infty_c(G)$ be an approximating sequence for $w^- \in \mathcal{D}(\mathcal{E}_V, G)$. Set $w_n := \varphi_n^+ \wedge w^-$. Hence $0 \leq w_n \in \mathcal{D}(\mathcal{E}_V, G)$, by Lemma A.1. Note that $w_n = w^- - (\varphi_n^+ - w^-)^-$. Therefore,

$$\begin{aligned} \mathcal{E}_V(w^- - w_n, w^- - w_n) &= \mathcal{E}_V((\varphi_n^+ - w^-)^-, (\varphi_n^+ - w^-)^-) \\ &\leq \mathcal{E}_V(\varphi_n - w^-, \varphi_n - w^-) \to 0. \end{aligned}$$

Thus (w_n) is a nonnegative approximating sequence for w^-. Since $w^+ \wedge w_n = 0$, we obtain

$$0 \leq \mathcal{E}_V(w, w_n) = -\mathcal{E}_V(w^-, w_n) \to -\mathcal{E}_V(w^-, w^-) \leq 0.$$

We conclude that $w^- = 0$. □

Remark A.5. Note that if $u \gneq 0$ is a super-solution to (A.8), then $-\Delta u \geq 0$ in G. Hence, by the weak Harnack inequality, $u > 0$ in G.

Remark A.6. If $\mathcal{E}_V$ satisfies the λ–property, then Lemmas A.3 and A.4 imply that equation (A.8) has a rich cone of positive super-solutions. Indeed, if $0 \lneq f \in \hat{\mathcal{D}}'(\mathcal{E}_V, G)$ and $\phi_* \in \mathcal{D}(\mathcal{E}_V, G)$ is the solution to (A.7), then $\phi_* > 0$ in G.

The following comparison principle is a straightforward consequence of Lemma A.4.

Corollary A.7. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $w \in H^1_{loc}(G)$ be a super-solution to* (A.8) *and $v \in H^1_{loc}(G)$ be a sub-solution to* (A.8) *such that $(w - v)^- \in \mathcal{D}(\mathcal{E}_V, G)$. Then $w \geq v$ in G.*

A version of the comparison principle below plays a crucial role in the analysis of asymptotic behavior of super-solutions to (3.1) in Section 4.

Lemma A.8. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $0 \leq w \in H^1_{loc}(G)$, $v \in \mathcal{D}(\mathcal{E}_V, G)$ and $w - v$ be a super-solution to equation* (A.8). *Then $w \geq v$ in G.*

Proof. Let (G_n) be an exhaustion of G, i.e. an increasing sequence of bounded smooth domains such that $G_n \Subset G_{n+1} \Subset G$ and $\cup G_n = G$. Note that $\lambda^{-1} \in L^\infty(G_n)$ and therefore $\mathcal{D}(\mathcal{E}_V, G_n) = H_0^1(G_n)$. Clearly $H_0^1(G_n)$ is a closed subspace of $\mathcal{D}(\mathcal{E}_V, G)$.

Let $v \in \mathcal{D}(\mathcal{E}_V, G)$. Let $f \in \mathcal{D}'(\mathcal{E}_V, G)$ be defined by

$$f(\varphi) := \mathcal{E}_V(v, \varphi), \qquad (\varphi \in \mathcal{D}(\mathcal{E}_V, G)).$$

By Lemma A.3 there exists the unique $v_n \in H_0^1(G_n)$ such that

$$\mathcal{E}_V(v_n, \varphi) = f(\varphi), \qquad \forall \varphi \in H_0^1(G_n).$$

Thus,

$$(-\Delta - V)(v - v_n) = 0 \quad \text{in } \ G_n,$$

and hence

$$(-\Delta - V)(w - v_n) \geq 0 \quad \text{in } \ G_n,$$

with $w - v_n \in H_{loc}^1(G_n)$ and $0 \leq (w - v_n)^- \leq v_n^+ \in H_0^1(G_n)$. Corollary A.7 implies that $v_n \leq w$ in G_n.

Let $\bar{v}_n$ denote the extension of v_n to G by zero. Clearly $\bar{v}_n \in \mathcal{D}(\mathcal{E}_V, G)$. To complete the proof it suffices to show that $\bar{v}_n \to v$ in $\mathcal{D}(\mathcal{E}_V, G)$. Indeed, by the construction of $\bar{v}_n$ we obtain

$$\mathcal{E}_V(\bar{v}_n, \bar{v}_n) = f(v_n) \leq \|f\|_{\mathcal{D}'} \|\bar{v}_n\|_{\mathcal{D}}.$$

Hence, the sequence $(\bar{v}_n)$ is bounded in $\mathcal{D}(\mathcal{E}_V, G)$. Thus there is a subsequence, which we still denote by $(\bar{v}_n)$, that converges weakly to $v_* \in \mathcal{D}(\mathcal{E}_V, G)$. Now let $\varphi \in H_c^1(G) \cap L_c^\infty(G)$. Then $\varphi \in H_0^1(G_n)$ for all $n \in \mathbb{N}$ large enough, and

$$\mathcal{E}_V(\bar{v}_n, \varphi) = f(\varphi).$$

Passing to the limit we conclude that

$$\mathcal{E}_V(v_*, \varphi) = f(\varphi), \qquad \forall \varphi \in H_c^1(G) \cap L_c^\infty(G),$$

and therefore $v_* = v$. Furthermore,

$$\mathcal{E}_V(\bar{v}_n - v, \bar{v}_n - v) = f(\bar{v}_n) - 2f(v) + f(v).$$

Since $f(\bar{v}_n) \to f(v)$, it follows that $\bar{v}_n \to v$ strongly in $\mathcal{D}(\mathcal{E}_V, G)$. □

Ground state transformation. If $\mathcal{E}_V$ satisfies the λ–property, then equation (A.8) has a rich cone of positive super-solutions, see Remark A.6. One can show that (A.8) has a positive solution if $\mathcal{E}_V$ is positive definite (but may not satisfy the λ–property, cf. [2, Theorem 3.1] and a detailed analysis in [30, 35]). Below we give a simple proof of the converse (cf. [2], [15] for the ground state transform, [4] for the Picone identity, [36] for the h–transform).

Lemma A.9. *Let $0 < \phi \in H^1_{loc}(G)$ be a (super-) solution to the equation*

$$(-\Delta - V)\phi = f \quad in \quad G, \tag{A.9}$$

where $0 \le f \in L^1_{loc}(G)$. Then the form $\mathcal{E}_V$ is positive definite in the sense of (A.3). *Moreover,*

$$\mathcal{D}(\mathcal{E}_V, G) \ni u \mapsto \frac{u}{\phi} \in H^1(G, \phi^2 dx) \tag{A.10}$$

and for all $u \in \mathcal{D}(\mathcal{E}_V, G)$ one has

$$\mathcal{E}_V(u,u)\,(\ge) = \int_G |\nabla\Big(\frac{u}{\phi}\Big)|^2 \phi^2 dx + \int_G u^2\, \frac{f}{\phi}\, dx. \tag{A.11}$$

Proof. Let $u \in H^1_c(G) \cap L^\infty_c(G)$. Then $\varphi := \frac{u^2}{\phi} \in H^1_c(G) \cap L^\infty_c(G)$. Testing (A.8) against φ we arrive at

$$2\int_G u\,\nabla u\,\frac{\nabla\phi}{\phi}\,dx\,(\ge) = \int_G u^2 \frac{|\nabla\phi|^2}{\phi^2}\,dx + \int_G Vu^2\,dx + \int_G \frac{f}{\phi} u^2\,dx.$$

Direct computation gives that

$$\begin{aligned}
&\int_G \big(|\nabla u|^2 - Vu^2\big)\,dx - \int_G |\nabla\Big(\frac{u}{\phi}\Big)|^2 \phi^2 dx \\
&= \int_G \Big(|\nabla u|^2 - Vu^2\Big)dx - \int_G \Big(\frac{|\nabla u|^2}{\phi^2} - 2u\,\nabla u\,\frac{\nabla\phi}{\phi^3} + u^2\frac{|\nabla\phi|^2}{\phi^4}\Big)\phi^2 dx \\
&= 2\int_G u\,\nabla u\,\frac{\nabla\phi}{\phi}\,dx - \int_G u^2\frac{|\nabla\phi|^2}{\phi^2}\,dx - \int_G Vu^2\,dx\,(\ge) = \int_G u^2\,\frac{f}{\phi}\,dx.
\end{aligned}$$

This proves (A.11) on $H^1_c(G) \cap L^\infty_c(G)$ and implies, in particular, that $\mathcal{E}_V(u,u) \ge 0$ on $H^1_c(G) \cap L^\infty_c(G)$. Therefore the extended Dirichlet space $\mathcal{D}(\mathcal{E}_V, G)$ is well defined.

Let $u \in \mathcal{D}(\mathcal{E}_V, G)$ and let $(\varphi_n) \subset H^1_c(G) \cap L^\infty_c(G)$ be an approximating sequence for u. Then $u_n := (-u) \vee \varphi_n \wedge u \in H^1_c(G) \cap L^\infty_c(G)$ is also an approximating sequence for u (cf. proof of Lemma A.4), and

$$0 \le \int_G |\nabla\Big(\frac{u_n}{\phi}\Big)|^2 \phi^2 dx \quad (\le) \;=\; \mathcal{E}_V(u_n, u_n) - \int_G u_n^2\,\frac{f}{\phi}\,dx$$

$$\to \qquad \mathcal{E}_V(u,u) - \int_G u^2 \, \frac{f}{\phi} \, dx.$$

Hence, the assertion follows by Fatou's lemma and, in the case of equality, by standard continuity arguments. □

The following straightforward corollary of Lemma A.9 is crucial in our analysis of nonexistence of positive solutions to the semilinear equation (2.1).

Corollary A.10. *Assume there exists* $u \in H^1_c(G) \cap L^\infty_c(G)$ *such that*

$$\mathcal{E}_V(u,u) < 0.$$

Then equation (A.8) *has no positive super-solution.*

Another interesting application of the ground state transformation is Barta's inequality.

Corollary A.11. *Assume that* $\mathcal{E}_V$ *is positive definite. Then for every* $0 < \phi \in H^1_{loc}(G)$ *such that* $(-\Delta - V)\phi \in L^1_{loc}(G)$ *the following inequality holds*

$$\inf_{x\in G} \frac{(-\Delta - V)\phi}{\phi} \leq \inf_{0\neq u \in C^\infty_c(G)} \frac{\mathcal{E}_V(u,u)}{\|u\|^2_{L^2}}. \tag{A.12}$$

Proof. Set $f := (-\Delta - V)\phi$. We may assume $f \geq 0$ in G (otherwise (A.12) is trivial). Then Lemma A.9 implies that

$$\int_G |\nabla u|^2 dx - \int_G V u^2 dx \geq \int_G u^2 \frac{f}{\phi} \, dx \geq \inf_{x\in G} \frac{f}{\phi} \int_G u^2 dx,$$

for all $u \in H^1_c(G) \cap L^\infty_c(G)$. So, the assertion follows. □

Remark A.12. Note that if $-\Delta - V$ admits a principal Dirichlet eigenfunction $\phi_1 > 0$ in G, then the equality in (A.12) is attained with $\phi = \phi_1$.

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