On the Log–Laplace equation for nonlocal operators generating sub–Markovian semigroups

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Abstract

We consider the semilinear Cauchy problem for a class of pseudo–differential operator generating sub–Markovian semigroup. Solutions of such problem with the negative definite nonlinearity play an important role in constructing branching measure–valued processes. We establish local existence and uniqueness of solutions in the context of the Dirichlet space associated to the problem. Comparison and global properties of solutions are also studied.

Keywords. Pseudo–differential operators, sub–Markovian semigroups, semilinear Cauchy problem, Dirichlet spaces, branching processes.

AMS subject classification. *Primary* 35S10; *secondary* 47D07, 60J80.

1 Introduction

We consider a class of pseudo differential operators

$$p(x,D)u(x) = (2\pi)^{-(N/2)} \int_{\mathbb{R}^N} e^{ix\xi} p(x,\xi)\hat{u}(\xi)d\xi$$

where $(x,\xi) \to p(x,\xi) \in \mathbb{R}$ is continuous symbol and for fixed $x \in \mathbb{R}^N$, the function $\xi \to p(x,\xi)$ is required to be negative definite in the sense of I. J. Schoenberg. Under suitable conditions the operator $p(x,D)$ extends from $C_0^\infty(\mathbb{R}^N)$ to a generator of sub–Markovian semigroup $(T_t)_{t\geq 0}$ on $L_2(\mathbb{R}^N)$.

In this note we are interested in the semilinear Cauchy problem for the operator $p(x,D)$

$$(1) \qquad \begin{cases} \frac{\partial u}{\partial t} + p(x,D)u = f(x,u) & \text{in} \quad [0,T) \times \mathbb{R}^N, \\ u(0,\cdot) = v_0 & \text{in} \quad \mathbb{R}^N, \end{cases}$$

here $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a given Carathéodory function, that is, $x \to f(x,u)$ is measurable for all $u \in \mathbb{R}$ and $u \to f(x,u)$ is continuous for almost all $x \in \mathbb{R}^N$.

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In the following we are interesting in the particular case when $u \to f(x,u)$ is a continuous negative definite function for almost all $x \in \mathbb{R}^N$. However we do not assume it explicitly up to the last section.

Solutions of the Cauchy problem (1) with the negative definite nonlinear part play an important role in constructing continuous state branching processes as introduced by M. Jirina [15, 16] and M. Motoo [21]. Often these problems are called Log–Laplace equations. We refer in particular to S. Watanabe [27] where we learned from the following result, see also the survey of D. Dawson [3] and a monograph of E. B. Dynkin [4]: in the case of continuous state branching processes related to Feller processes on a compact state space it is known that these processes are in one–to–one correspondence to Ψ–semigroups and the solutions of (1) will give Ψ–semigroups. Thus in order to construct continuous state branching processes it is helpful to solve (1). Note that there is a very large literature on branching processes and measure–valued processes where solutions to equation (1) are used, we refer in addition to the papers already mentioned the very general considerations due to P. J. Fitzsimmons [6], and the paper [17] of N. Konno and T. Shiga in which relations to stochastic partial differential equations are discussed. However, in all papers mentioned and other paper known to us, equation (1) is studied (by standard methods, see A. Pazy [24]) in the frame of spaces of bounded functions. In this case no problem with the growth of nonlinearity show up, provided it is locally Lipschitzian. Our results give solutions to (1) in the context of the Dirichlet space associated to the operator $-p(x,D)$. This causes serious problems with the growth of the nonlinearity: We run to the critical exponent problem. But the value of the critical exponent is now determined by the L_2–domain of the operator $-p(x,D)$, not by the Laplacian. This fact makes our discussion quite different from the related papers working in spaces of bounded functions. Our Theorem 1 is a result which takes the appearance of a critical exponent in a straight forward way into account. In Theorem 2 we use explicitly the sub–Markovian property of the semigroup generated by $-p(x,D)$ to extend the class of nonlinearities for which an existence result holds. Our aim in a further investigation is to construct new branching processes in the context of Dirichlet form by using the solvability of (1) in L_2–spaces.

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2 A class of Pseudo Differential Operators

Let $\psi : \mathbb{R}^N \to \mathbb{R}$ be a fixed continuous negative definite function, i.e. ψ is continuous, $\psi(0) \geq 0$ and $\xi \to e^{-t\psi(\xi)}$ is for all $t \geq 0$ positive definite. We assume in addition that

$$\psi(\xi) \geq c|\xi|^{2r} \quad \text{for } |\xi| \geq R \tag{2}$$

holds for some $r \in (0,1]$ and $R, c > 0$.

We define the anisotropic Sobolev space $H^{\psi,s}(\mathbb{R}^N)$, $s \geq 0$ by

$$H^{\psi,s}(\mathbb{R}^N) = \{u \in L_2(\mathbb{R}^N) : \|u\|_{\psi,s} < \infty\},$$

where

$$\|u\|^2_{\psi,s} = \int_{\mathbb{R}^N} (1 + \psi(\xi))^s |\hat{u}(\xi)|^2 d\xi,$$

giving the scalar product

$$(u, v)_{\psi,s} = \int_{\mathbb{R}^N} (1 + \psi(\xi))^s \hat{u}(\xi)\overline{\hat{v}(\xi)} d\xi.$$

For $\psi(\xi) = |\xi|^2$ the space $H^{\psi,s}(\mathbb{R}^N)$ coincides with the usual Sobolev space $H^s(\mathbb{R}^N)$ with the norm and scalar product denoted by $\|\cdot\|_s$ and $(\cdot,\cdot)_s$. Clearly, (2) implies continuous embedding $H^{\psi,s}(\mathbb{R}^N) \subseteq H^{rs}(\mathbb{R}^N)$. Using embeddings theorems for the Sobolev scale $H^{rs}(\mathbb{R}^N)$ we can obtain embeddings results for $H^{\psi,s}(\mathbb{R}^N)$.

In the following we will consider pseudo–differential operator $p(x, D)$ having a symbol $(x, \xi) \to p(x, \xi) \in \mathbb{R}$ which is continuous and for fixed $x \in \mathbb{R}^N$, the function $\xi \to p(x, \xi)$ is required to be negative definite. Further we assume that $-p(x, D)$ extends to a symmetric Dirichlet operator A with domain $H^{\psi,2}(\mathbb{R}^N)$. It is well known that A generates a symmetric sub–Markovian semigroup $(T_t)_{t\geq 0}$ on $L_2(\mathbb{R}^N)$ and in fact, due to a result of E.M. Stein [25] this semigroup is analytic. Often we will write $-p(x, D)$ instead of A.

We refer to the paper [10] and [11] of W. Hoh and to the papers [12, 13] of the first named author where a lot of concrete examples of conditions on $p(x, \xi)$ are given which implies our assumptions. Note that in particular any continuous negative definite function $\psi : \mathbb{R}^N \to \mathbb{R}$ gives rise to a pseudo–differential operator $-\psi(D)$ on $H^{\psi,2}(\mathbb{R}^N)$ extending to a generator of a symmetric sub–Markovian semigroup on $L_2(\mathbb{R}^N)$.

3 Local existence and uniqueness

In the following we will always assume, that the function $u \to f(x, u)$ is locally Lipschitzian for almost all $x \in \mathbb{R}^N$, i.e. there exist Carathéodory function $\phi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that

$$|f(x, u) - f(x, v)| \leq \phi(x, w)|u - v| \quad \text{for } |u|, |v| \leq w. \tag{3}$$

Note that (3) implies the growth estimate

$$|f(x, u)| \leq \phi(x, u)|u| + f(x, 0). \tag{4}$$

Let

$$f(u)(x) = f(x, u(x))$$

be the superposition operator generated by $f(x,u)$. We rewrite (1) as an abstract Cauchy problem

$$\left\{\begin{array}{l} \frac{du}{dt} = Au + f(u), \\ u(0) = v_0, \end{array}\right. \tag{5}$$

where A is a generator of sub–Markovian, and hence, analytical semigroup in $L_2(\mathbb{R}^N)$.

Let us recall several basic notions from the theory of semilinear Cauchy problems for generators of analytical semigroups, see [9] as a basic reference, see also [24, 23]. A *(classical) solution* on $(0,T)$ of the Cauchy problem (5) is a continuous function $u : [0,T) \to L_2(\mathbb{R}^N)$ such that

$$u(0) = v_0, \quad u(t) \in H^{\psi,2}(\mathbb{R}^N) \quad \text{for } \ t \in (0,T),$$

$u \in \mathcal{C}^1((0,T); L_2(\mathbb{R}^N))$ and the pseudo differential equation (5) is satisfied on $(0,T)$. A solution u on $(0,T_{max})$ is *maximal*, if there is no solution of (5) on $(0,T)$ for $T > T_{max}$. A solution u is *global*, if $T_{max} = +\infty$. A solution u *blows–up in* $H^{\psi,s}(\mathbb{R}^N)$, if $T_{max} < +\infty$ and

$$\lim_{t\to T_{max}} \|u(t_n)\|_{\psi,s} = +\infty.$$

We say that solutions of (5) *continuously depends on the initial data in* $H^{\psi,s}(\mathbb{R}^N)$ if $v_n \to v_0$ in $H^{\psi,s}(\mathbb{R}^N)$ implies that for any $T < T_{max}(v_0)$ the solution u_n of (5) with $u_n(0) = v_n$ exists for $t \in [0,T]$ if n sufficiently large and $u_n \to u$ uniformly in $\mathcal{C}([0,T], H^{\psi,s}(\mathbb{R}^N))$.

If u is a classical solution of (5) on $(0,T)$ then u solves the Volterra–type nonlinear integral equation

$$u(t) = T_t v_0 + \int_0^t T_{t-s} f(u(s))\, ds, \quad t \in [0,T). \tag{6}$$

Now if $u \in \mathcal{C}((0,T); L_2(\mathbb{R}^N))$ is a solution of (6) then u is called a *mild* solution of the Cauchy problem (5). Under certain regularity conditions a mild solution is also a classical solution of (5).

Let $s \geq 0$. It follows from (2) and the Sobolev embedding theorem that we have the sequence of continuous embeddings

$$H^{\psi,s}(\mathbb{R}^N) \subseteq H^{rs}(\mathbb{R}^N) \subset L_{2\wedge s^*}(\mathbb{R}^N) \subset L_2(\mathbb{R}^N), \tag{7}$$

here $L_{2\wedge s^*}(\mathbb{R}^N)$ is the space $L_2(\mathbb{R}^N) \cap L_{s^*}(\mathbb{R}^N)$ equipped with norm

$$\|u\|_{L_{2\wedge s^*}} = \max\left\{\|u\|_0, \|u\|_{L_{s^*}}\right\},$$

and

$$s^* = \left\{\begin{array}{ll} \frac{2N}{N-2rs}, & 2rs < N, \\ \text{any } p \geq 2, & 2rs \geq N, \end{array}\right.$$

is a critical Sobolev exponent. If $2rs > N$ then $H^{\psi,s}(\mathbb{R}^N) \subset L_{2\wedge\infty}(\mathbb{R}^N)$ and one can put $s^* = +\infty$. However we do not need this fact in following.

Theorem 1 *Assume that for some $s \in [0,2)$ the estimate*

$$\phi(x,w) \le a(|w|^{\frac{s^*}{2}-1}+1) \tag{8}$$

holds with $a>0$ and $f(x,0) \in L_2(\mathbb{R}^N)$. Then for every $v_0 \in H^{\psi,s}(\mathbb{R}^N)$:
i) there exist $T_{max} = T_{max}(v_0) > 0$ such that the Cauchy problem (5) has a unique maximal classical solution $u \in \mathcal{C}((0,T_{max}), H^{\psi,s}(\mathbb{R}^N))$;
ii) either u is a global solution or else the solution u blows–up in $H^{\psi,s}(\mathbb{R}^N)$;
iii) if $s=0$ then u is a global solution;
iv) the solutions of (5) depends continuously on the initial data in $H^{\psi,s}(\mathbb{R}^N)$.

Proof. To prove *(i)* we will apply the local existence theorem for semilinear problems related to generators of analytical semigroups [9, Theorem 3.3.3, p.54]. For this, let us rewrite (5) as

$$\begin{cases} \frac{du}{dt} = (A-I)u + u + f(u), \\ u(0) = v_0. \end{cases} \tag{9}$$

Since A is a symmetric Dirichlet operator, $-A$ is nonnegative definite on $L_2(\mathbb{R}^N)$. Hence $0 \in \rho(A-I)$ and for $s \in [0,2]$ the domain of the fractional powers $(I-A)^{s/2}$ are obtained by complex interpolation which yields $H^{\psi,s}(\mathbb{R}^N)$ is the domain of $(I-A)^{s/2}$. Thus to prove the existence of unique classical solution of (5) on $(0,T)$ it remains to show that the mapping $u+f(u)$ is a locally Lipschitzian as a mapping from $H^{\psi,s}(\mathbb{R}^N)$ into $L_2(\mathbb{R}^N)$. Taking into account the continuous embeddings (7) it is sufficient to prove that f is a locally Lipschitzian as an operator from $L_{2\wedge s^*}(\mathbb{R}^N)$ into $L_2(\mathbb{R}^N)$.

For $u,v \in L_{2\wedge s^*}(\mathbb{R}^N)$ and $w = |u| \vee |v| \in L_{2\wedge s^*}(\mathbb{R}^N)$ by (8) and Hölder inequality we get

$$\begin{aligned} \int_{\mathbb{R}^N} |f(x,u)-f(x,v)|^2 dx &\le \int_{\mathbb{R}^N} |\phi(x,w)|^2 |u-v|^2\, dx \\ &\le a^2 \int_{\mathbb{R}^N} (|w|^{\frac{s^*}{2}-1}+1)^2 |u-v|^2\, dx \\ &\le 2a^2 \int_{\mathbb{R}^N} |w|^{s^*-2}|u-v|^2\, dx + 2a^2 \int_{\mathbb{R}^N} |u-v|^2\, dx \\ &\le 2a^2 \|w\|_{L_{s^*}}^{s^*-2} \|u-v\|_{L_{s^*}}^2 + 2a^2 \|u-v\|_0^2 \end{aligned}$$

Thus

$$\|f(u)-f(v)\|_0 \le \sqrt{2}a(1+\|w\|_{L_{2\wedge s^*}}^{s^*-2})^{1/2} \|u-v\|_{L_{2\wedge s^*}}$$

and

$$\|u\|_{L_{2\wedge s^*}}, \|v\|_{L_{2\wedge s^*}} \le \|w\|_{L_{2\wedge s^*}}.$$

This means that f is locally Lipschitzian from $L_{2\wedge s^*}(\mathbb{R}^N)$ into $L_2(\mathbb{R}^N)$ and hence there exist unique classical solution of (5) on $[0,T)$.

To construct a maximal solution of (5) suppose that $\|u(t)\|_{\psi,s}$ is bounded for all $t \in [0,T)$. Note that condition (8) implies that the superposition operator f

maps bounded sets from $L_{2\wedge s^*}(\mathbb{R}^N)$, and hence from $H^{\psi,s}(\mathbb{R}^N)$ into the bounded sets in $L_2(\mathbb{R}^N)$, cf. [1]. Then by [9, Theorem 3.3.4, p.55], there exist $u_1 \in H^{\psi,s}(\mathbb{R}^N)$ such that $u(t) \to u_1$ in $H^{\psi,s}(\mathbb{R}^N)$ as $t \to T$ which implies the solution $u(t)$ may be extended beyond time T. In such a way one can extend the solution $u(t)$ while $\|u(t)\|_{\psi,s}$ remains bounded obtaining a global solution of (5). Otherwise $u(t)$ is unbounded in $H^{\psi,s}(\mathbb{R}^N)$ on some finite interval $(0, T_{max})$ which also proves *(ii)*.

iii) In the case $s = 0$ the condition (8) implies that the function $\phi(x, w)$ is bounded. Then the superposition operator f is globally Lipschitzian on the space $H^{\psi,0}(\mathbb{R}^N) = L_2(\mathbb{R}^N)$. It is known that in this case there exist global classical solution of (5), cf. [23, Theorem 5.8, p.215].

iv) The continuous dependence of solutions on the initial data in $H^{\psi,s}(\mathbb{R}^N)$ follows immediately from [9, Theorem 3.4.1, p.62] since the operator f is locally Lipschitzian on $H^{\psi,s}(\mathbb{R}^N)$. □

Remark 1 If the function $u \to f(x, u)$ is differentiable for almost all $x \in \mathbb{R}^N$, $f(x, 0) \in L_2(\mathbb{R}^N)$ and the estimate

$$(10) \qquad |f'_u(x,u)| \le a(|u|^{\frac{s^*}{2}-1} + 1)$$

holds for some $a > 0$ then the superposition operator f is continuously differentiable as operator from $L_{2\wedge s^*}(\mathbb{R}^N)$, and hence, from $H^{\psi,s}(\mathbb{R}^N)$ into $L_2(\mathbb{R}^N)$. Its Fréchet derivative given by the formula

$$f'(u)(x) = f'_u(x, u(x)),$$

cf. [1]. In particular, in this case $u \to f(x, u)$ is locally Lipschitzian for almost all $x \in \mathbb{R}^N$ and condition (8) always holds.

Remark 2 The condition (3) can not be relaxed. From [1, Theorem 5.5, p.158] it follows that if the superposition operator f is locally Lipschitzian as operator from $L_{2\wedge s^*}(\mathbb{R}^N)$ into $L_2(\mathbb{R}^N)$ then the function $u \to f(x, u)$ is locally Lipschitzian for almost all $x \in \mathbb{R}^N$.

Remark 3 The condition (8) implies by (4) that the function $f(x, u)$ satisfies the growth estimate

$$(11) \qquad |f(x,u)| \le a(|u|^{\frac{s^*}{2}} + |u|) + f(x,0)$$

for some $a > 0$ with $f(x, 0) \in L_2(\mathbb{R}^N)$.

Remark 4 The statements *(i, iv)* imply that the Cauchy problem (5) generates a local semi–flow in $H^{\psi,s}(\mathbb{R}^N)$.

Using the sub–Markovian property of the semigroup $(T_t)_{t\ge 0}$ the restrictions on the growth of $\phi(x, u)$ and $f(x, u)$ in the case $N < 2rs$ can be avoided at least for the initial data $v_0 \in L_{2\wedge\infty}(\mathbb{R}^N)$.

Theorem 2 *Assume that*

$$\sup_{|w|\le r} \phi(x,w) \in L_\infty(\mathbb{R}^N) \quad \text{for all } r \ge 0 \tag{12}$$

and $f(x,0) \in L_{2\wedge\infty}(\mathbb{R}^N)$. *Then for every* $v_0 \in L_{2\wedge\infty}(\mathbb{R}^N)$:
i) there exist $T_{max} = T_{max}(v_0) > 0$ *such that the Cauchy problem (5) has a unique maximal classical solution* u *and* $u(t) \in L_{2\wedge\infty}(\mathbb{R}^N)$ *for* $t \in [0, T_{max})$;
ii) either $T_{max} = +\infty$ *or else the solution* u *blows–up in* $L_{2\wedge\infty}(\mathbb{R}^N)$.
iii) the solutions of (5) depend continuously on the initial data in $L_2(\mathbb{R}^N)$.

Proof. Let

$$m = \|v_0\|_{L_\infty} + 1.$$

Define a truncation of the nonlinearity $f(x,u)$ by means of formula

$$\tilde{f}(x,u) = \begin{cases} f(x,-m), & \text{if} \quad u < -m, \\ f(x,u), & \text{if} \quad -m \le u \le m, \\ f(x,m), & \text{if} \quad u > m. \end{cases}$$

The truncation $\tilde{f}(x,u)$ is a bounded Carathéodory function satisfying condition (3) with

$$\tilde{\phi}(x,w) = \sup_{|w|\le m} \phi(x,w) \le a_m < +\infty.$$

By (4) this implies the uniform bound

$$|\tilde{f}(x,u)| \le a = m a_m + \|f(x,0)\|_{L_\infty}. \tag{13}$$

Applying Theorem 1 with $s = 0$ we obtain the existence of a unique global solution $\tilde{u}$ of the truncated problem

$$\begin{cases} \frac{du}{dt} = Au + \tilde{f}(u), \\ u(0) = v_0. \end{cases} \tag{14}$$

Since $\tilde{u}$ is also a mild solution of (14) and using the sub–Markovian property of the semigroup $(T_t)_{t\ge 0}$ and (13) we derive

$$\|\tilde{u}(t)\|_{L_\infty} \le \|T_t v_0\|_{L_\infty} + \int_0^t \|T_{t-s}\tilde{f}(\tilde{u}(s))\|_{L_\infty}\, ds \le$$

$$\|v_0\|_{L_\infty} + \int_0^t \|\tilde{f}(\tilde{u}(s))\|_{L_\infty}\, ds \le \|v_0\|_{L_\infty} + at, \qquad t \in [0,+\infty).$$

This implies

$$\|\tilde{u}(t)\|_{L_\infty} \le m \quad \text{for } t \in [0, 1/a)$$

and hence

$$\tilde{f}(\tilde{u}(t)) = f(\tilde{u}(t)) \quad \text{for } t \in [0, 1/a).$$

This means that $\tilde{u}$ is a local solution of the original problem (5) on $(0, 1/a)$ and also $u(t) \in L_{2\wedge\infty}(\mathbb{R}^N)$ for $(0, 1/a)$;

To construct a maximal solution of (5) suppose that $\|u(t)\|_0$ is bounded for all $t \in [0,T)$. Note that truncated superposition operator $\tilde{f}$ is bounded as operator in $L_2(\mathbb{R}^N)$, cf. [1]. Then by [9] there exist $u_1 \in L_2(\mathbb{R}^N)$ such that $u(t) \to u_1$ in $L_2(\mathbb{R}^N)$ as $t \to T$. But $u(t) \in L_{2\wedge\infty}(\mathbb{R}^N)$ for $t \in [0,T)$ and $L_{2\wedge\infty}(\mathbb{R}^N)$ is a closed subspace of $L_2(\mathbb{R}^N)$. Hence the L_2–limit $u_1 \in L_{2\wedge\infty}(\mathbb{R}^N)$ and using, if necessary, new truncation $\tilde{f}_1(x,u)$, the solution $u(t)$ may be extended beyond time T. In such a way one can extend the solution $u(t)$ in $L_{2\wedge\infty}(\mathbb{R}^N)$ while $\|u(t)\|_0$ remains bounded obtaining a global solution of (5). Otherwise $u(t)$ is unbounded in $L_2(\mathbb{R}^N)$ on the finite interval $(0,T_{max})$ which also proves *(ii)*.

iii) The continuous dependence of solutions on the initial data in $L_2(\mathbb{R}^N)$ follows immediately from Theorem 1, iv) since for any $T < T_{max}(v_0)$ the solution u on $[0,T]$ solves a suitably truncated problem which satisfies (8) with $s = 0$. □

Remark 5 The condition (12) implies that for each $r \geq 0$ there exists $a_r > 0$ such that

$$|f(x,u)| \leq a_r|u| + f(x,0) \qquad \text{for } |u| \leq r \tag{15}$$

with $f(x,0) \in L_{2\wedge\infty}(\mathbb{R}^N)$. This means that $f(x,u)$ admits an arbitrary fast growth in u "at infinity".

4 Positivity preserving and comparison

Using the fact that the semigroup $(T_t)_{t\geq 0}$ generated by A is sub–Markovian we may prove that the solution to the initial–value problem (5) is positive provided v_0 is positive and $f(x,u) \geq 0$ holds. More precisely we have the following comparison result.

Theorem 3 *Assume that*

$$f(x,u) \geq 0 \tag{16}$$

and let u be a maximal classical solution of the problem (5) on $[0,T_{max})$. Then

$$u(t) \geq T_t v_0 \quad \text{for } t \in [0,T_{max}). \tag{17}$$

In particular, if $v_0 \geq 0$, then $u(t) \geq 0$ for $t \in [0,T_{max})$.

Proof. Since u is a classical solution, u is also a mild solution, i.e. u solves (6). By (16) and since $(T_t)_{t\geq 0}$ is a sub–Markovian semigroup, in particular, $(T_t)_{t\geq 0}$ is positivity preserving we have

$$\int_0^t T_{t-s} f(u(s))\, ds \geq 0, \qquad t \in [0,T_{max})$$

Then

$$u(t) = T_t v_0 + \int_0^t T_{t-s} f(u(s))\, ds \geq T_t v_0, \qquad t \in [0,T_{max}).$$

Again, since $(T_t)_{t\geq 0}$ is positivity preserving we have $u(t) \geq 0$ for $t \in [0,T_{max})$ if $v_0 \geq 0$. □

Remark 6 Theorem 3 can be used to prove other types of comparison results, or to develop a certain sub– and super–solutions technique for the problem (5), cf. [22] for various applications of such results in the local case $A = \Delta$ and [5] for relations with branching processes.

5 Global existence and blow–up

A first global existence result is the following theorem.

Theorem 4 *Assume that for some $s \in [0, 2)$ condition (8) holds and*

$$|f(x, u)| \leq a|u| + b(x) \tag{18}$$

for some $a > 0$ and $b \in L_2(\mathbb{R}^N)$. Then for every $v_0 \in H^{\psi,s}(\mathbb{R}^N)$ the Cauchy problem (5) has a unique global classical solution.

Proof. The existence of the unique maximal solution proved in Theorem 1. To prove the global existence it is enough to get for some $c > 0$ a uniform estimate

$$\|f(u(t))\|_0 \leq c(1 + \|u(t)\|_{\psi,s}), \tag{19}$$

in the existence interval of the solution, see cf. [9, Corollary 3.3.5, p.56].

By (18) for $t \in (0, T_{max})$ we obtain

$$\|f(u(t))\|_0^2 \leq \int_{\mathbb{R}^N} (a|u(t)| + b(x))^2 \, dx \leq$$

$$2a^2 \int_{\mathbb{R}^N} |u(t)|^2 dx + 2 \int_{\mathbb{R}^N} |b(x)|^2 dx \leq a_1^2 \|u(t)\|_{\psi,s}^2 + 2\|b\|_0^2,$$

which implies (19). □

Remark 7 The statement of this theorem does not follow from Theorem 1, iii) since (18) does not imply that $u \to f(x, u)$ is globally Lipschitzian for a.e. $x \in \mathbb{R}^N$.

By a standard arguments, cf. [7], one can show that if $f(x, u) \geq 0$ has a superlinear growth, the solution of the problem (5) may really blow–up. More precisely we have the following

Theorem 5 *Assume that $\lambda_1 = \inf \sigma(-A)$ is an eigenvalue of $-A$ and that the corresponding eigenfunction e_1 belongs to $H^{\psi,2}(\mathbb{R}^N) \cap L_1(\mathbb{R}^N)$. Assume also that $f(x, u) = f(u)$ is a convex smooth function,*

$$f(0) = 0, \quad f(u) \geq 0, \quad f'(u) \geq 0 \qquad \text{for } u \geq 0, \tag{20}$$

$$\int_1^{+\infty} \frac{du}{f(u)} < \infty, \tag{21}$$

and for some $s \in [0, 2)$ condition (8) holds. Then there exists $v_0 \in H^{\psi,s}(\mathbb{R}^N)$, $v_0 \geq 0$ such that the solution u of the problem (5) blows–up in $L_2(\mathbb{R}^N)$.

Proof. By Theorem 1 there exist a maximal solution of problem (5) defined on $[0, T_{max})$. By Theorem 3 we have $u(t) \geq 0$ for $t \in [0, T_{max})$ since $v_0 \geq 0$.

Consider the function

$$\gamma(t) = \int_{\mathbb{R}^N} e_1\, u(t)\, dx$$

where we assume that $\|e_1\|_{L_1} = 1$. Then

$$\gamma'(t) = \int_{\mathbb{R}^N} e_1\, \frac{d}{dt} u(t)\, dx = \int_{\mathbb{R}^N} e_1\, (Au(t) + f(u(t)))\, dx.$$

Note that $e_1 \geq 0$ since A is a Dirichlet operator. Since $e_1 \in L_1(\mathbb{R}^N)$, $e_1 \geq 0$ and $f(u)$ is convex applying Jensen inequality we have

$$\int_{\mathbb{R}^N} f(u(t))\, (e_1 dx) \geq f\left(\int_{\mathbb{R}^N} u(t)\, (e_1 dx)\right)$$

and since e_1 is an eigenfunction of A we get

$$\gamma'(t) \geq -\lambda_1 \int_{\mathbb{R}^N} e_1\, u(t)\, dx + f(\int_{\mathbb{R}^N} e_1\, u(t)\, dx) = -\lambda_1 \gamma(t) + f(\gamma(t)).$$

This implies that $\lim_{t \to T_{max}} \gamma(t) = +\infty$ if we choose $\|v_0\|_0$ sufficiently large. Since $u(t) \geq 0$ for $t \in [0, T_{max})$ this means that u blows–up in $L_2(\mathbb{R}^N)$. □

Remark 8 Actually the eigenfunctions e_k of A (if exist) do always belong to $H^{\psi,2}(\mathbb{R}^N) \cap L_{1\wedge\infty}(\mathbb{R}^N)$. To show this, at first, let us note that by the spectral theorem the generator $-A$ and the semigroup operator T_t for any $t > 0$ have the same eigenfunctions. By embeddings (7) we already know that the domain of the sub–Markovian generator $-A$ is contained in $L_{s^*}(\mathbb{R}^N)$ with $s^* > 2$. Then, by the result taken from Varopoulos et al. [26], for any $t > 0$ and $1 \leq p < q \leq \infty$ the operator T_t maps $L_p(\mathbb{R}^N)$ into $L_q(\mathbb{R}^N)$. In particular, T_t maps $L_2(\mathbb{R}^N)$ into $L_\infty(\mathbb{R}^N)$ and hence $e_k \in L_\infty(\mathbb{R}^N)$. Further, T_t maps $L_1(\mathbb{R}^N)$ into $L_\infty(\mathbb{R}^N)$, and, by duality, also $L_\infty(\mathbb{R}^N)$ into $L_1(\mathbb{R}^N)$. Hence e_k belongs to $L_{1\wedge\infty}(\mathbb{R}^N)$.

6 Log–Laplace equation

Now let us consider the adaptation of our result to the case of Log–Laplace equation, i.e. to the problem

$$\begin{cases} \frac{du}{dt} = Au + f(u), \\ u(0) = v_0, \end{cases} \tag{22}$$

with an operator A as before and an operator f generated by a Carathéordory function $f(x, u)$ with the property that $u \to f(x, u)$ is negative–definite for almost all $x \in \mathbb{R}^N$.

First, let us note that $f(x, u) \geq 0$ by the properties of negative–definite functions. Hence Theorem 3 implies that a solution $u(t)$ of (22) with $u(0) =$

$v_0 \geq 0$ remains nonnegative on the entire interval of existence, so we have the positivity–preserving property for the Log–Laplace equation. On the other hand the Log–Laplace equations inherit many difficulties known from well studied local case $A = \Delta$, including supercritical growth of nonlinearity, nonglobal existence and blow–up, nonuniqueness. Let us consider several examples.

Example 1 Consider the functions

$$f_1(x,u) = a(x)\log(1+u^2) + b(x),$$

$$f_2(x,u) = \frac{a(x)u^2}{1+u^2} + b(x),$$

$$f_3(x,u) = \sqrt{u^2 + b^2(x)}, \qquad f_4(x,u) = 1 - \cos(u)$$

with $a \in L_\infty(\mathbb{R}^N)$, $b \in L_2(\mathbb{R}^N)$ and $a, b \geq 0$.

It is known that these functions are negative definite in u for almost all $x \in \mathbb{R}^N$, cf. [2]. Moreover, f_i are smooth functions in u, their derivatives are uniformly bounded in x and $f_i(x,0) \in L_2(\mathbb{R}^N)$. By (10) in all these cases Theorem 1 holds with $s = 0$ and the Log–Laplace equation for f_i has a unique global solution for all initial data $v_0 \in L_2(\mathbb{R}^N)$.

Example 2 Now let us consider the function

$$f_{(\alpha)}(x,u) = a(x)|u|^\alpha + b(x)$$

with $a \in L_\infty(\mathbb{R}^N)$, $b \in L_2(\mathbb{R}^N)$ and $a, b \geq 0$. If $\alpha \in (0,2]$ the function $f_{(\alpha)}$ is negative definite in u for almost all $x \in \mathbb{R}^N$. However in this case the situation is more complicated. One can distinguish the following cases.

a) Supercritical growth: $\alpha \in (2 \wedge s^*/2, 2]$.

In this case we can not apply Theorem 1 because of "supercritical growth" of $f_{(\alpha)}$. Using Theorem 2 we can establish the existence of maximal solution for (22) with initial data $v_0 \in L_{2\wedge\infty}(\mathbb{R}^N)$ under the additional assumption $b \in L_\infty(\mathbb{R}^N)$. By Theorem 5 this solution may blow–up in finite time.

b) Subcritical growth: $\alpha \in (1, 2 \wedge s^*/2]$.

In this case using Theorem 1 we can establish the existence of unique maximal solution for (22) with initial data $v_0 \in H^{\psi,s}(\mathbb{R}^N)$. By Theorem 5 this solution may blows-up in finite time.

c) Case $\alpha = 1$.

In this case applying Theorem 1 with $s = 0$ we establish the existence of unique global solution for (22) for all initial data $v_0 \in L_2(\mathbb{R}^N)$.

d) Nonlipschitzian case: $\alpha \in (0,1)$.

In this case the function $f_{(\alpha)}$ do not satisfy the local Lipschitz condition. Using our methods we can not say anything about solvability of (22). The additional difficulty in this case is that the operator $f_{(\alpha)}$ does not map $H^{\psi,s}(\mathbb{R}^N)$ into $L_2(\mathbb{R}^N)$ for any $s \in (0,1)$. This is because of the superlinear growth of $u \to f_{(\alpha)}(x,u)$ near zero.

Remark 9 In the case $A = \Delta$ and $f_{(\alpha)}(x,u) = |u|^\alpha$ with $\alpha \in (0,1)$ using the method of monotone iterations one can establish the existence of unique global solution of (5) for nonnegative initial data $v_0 \in L_2(\mathbb{R}^N)$, $v_0 \geq 0$, $u \neq 0$, cf. [22]. However the uniqueness for this problem really fails at zero. More precisely a certain extinction phenomenon takes place. The global solution u of (5) may vanish after finite extinction time $T_{ext} < +\infty$, i.e. $u(t) = 0$ for $t \geq T_{ext}$, cf. [8, 19] and the references therein. One can expect that a similar result could be obtained for the problem (5) also in the general case of operators A generated by $-p(x,D)$.

Remark 10 Finally let us consider the stationary Log–Laplace equation

$$Au + f(u) = 0, \qquad u \in H^{\psi,s}(\mathbb{R}^N). \tag{23}$$

Solutions of this problem are stationary orbits of a (local) semi–flow associated with the Cauchy problem (22).

Suppose in addition that $f(x,0) = 0$. Then $u = 0$ is a trivial solution of (23). One can show that (23) has no nonpositive nontrivial solutions. On the other hand in the local case $A = \Delta - V(x)$ there is a huge literature about existence (and also non–existence) of multiple nontrivial positive solutions for (23), see cf. [20, 18]. It seems that at least some part of these results also hold in some form in the general case of nonlocal A generated by $-p(x,D)$, see [14] for some preliminary results in this direction.

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