On the Semilinear Dirichlet Problem for nonlocal operators generating symmetric Dirichlet forms

Niels Jacob* Vitaly Moroz†

1 Introduction

We consider a class of pseudo differential operators

$$p(x,D)u(x) = (2\pi)^{-(N/2)} \int_{\mathbb{R}^N} e^{ix\xi} p(x,\xi)\hat{u}(\xi)d\xi \tag{1}$$

where $p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a real valued continuous symbol such that $p(x,\cdot) : \mathbb{R}^N \to \mathbb{R}$ is negative definite in the sense of I. J. Schoenberg. Under suitable conditions $p(x,D)$ extends from $C_0^\infty(\mathbb{R}^N)$ to a generator of a symmetric Dirichlet form $(B, D(B))$ with domain $D(B) \subset L_2(\mathbb{R}^N)$ and

$$B(u,v) = (p(x,D)u,v)_{L_2} \quad \text{for} \quad u,v \in C_0^\infty(\mathbb{R}^N).$$

In this paper we are interested in the semilinear boundary value problems for $p(x,D)$ on some open set $\Omega \subset \mathbb{R}^N$. The main difficulty which arise is that $p(x,D)$ is in general a non–local operator not satisfying the transmission condition. Nonlocality means that $supp(u) \subseteq \Omega'$ does not imply $supp(p(x,D)u) \subseteq \Omega'$ for all open sets $\Omega' \subseteq \mathbb{R}^N$. Several approaches to the linear boundary value problems for the operator $p(x,D)$ were considered in the papers [6, 10], see also [9]. From the considerations in these papers it seems to be reasonable to give the following formulation of the semilinear Dirichlet problem for $p(x,D)$ on Ω: *given a Caratheodory function* $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, *find* $u : \mathbb{R}^N \to \mathbb{R}$ *such that*

$$\begin{cases} p(x,D)u = f(x,u) & \text{a.e. in } \Omega, \\ u = 0 & \text{a.e. in } \Omega^c, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is an open bounded set with sufficiently smooth boundary $\partial\Omega$ and complement $\Omega^c = \mathbb{R}^N \setminus \Omega$.

We will use a classical variational approach to handle a weak formulation of this problem. According to such approach the weak solutions of the problem

*Institut für Theoretische Informatik und Mathematik, Universität der Bundeswehr, Werner Heisenberg–Weg 39, 85577 Neubiberg, Germany

†Mechanics and Mathematics Department, Belorussian State University, Skariny av. 4, 220080 Minsk, Belarus; *currently* Institut für Theoretische Informatik und Mathematik, Universität der Bundeswehr, Werner Heisenberg–Weg 39, 85577 Neubiberg, Germany

(2) correspond to the minima of the energy functional for (2) on the certain anisotropic Sobolev space. The existence of the minima for the energy functional is provided by the usual one–sided estimates for the nonlinearity. Further we will see that the property of being a Dirichlet form enables us to develop truncation techniques for the problem (2) which seems to be new for nonlocal pseudo differential operators. Such techniques, typical for the second–order partial differential operators allow to obtain some results about existence of bounded positive solutions for the problem (2) and to relax the growth conditions on the nonlinearity. The motivation for considering such a problem is given by the theory of superprocesses, see E. B. Dynkin [4]. We will come back to these relations in another paper.

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2 A class of Pseudo Differential Operators

Let us recall some results from [7], see also [9]. Let $a^2 : \mathbb{R}^N \to \mathbb{R}$ be a real valued continuous negative definite function, that is a^2 is a continuous function such that $a^2(0) \geq 0$ and for all $t > 0$ the function $\xi \to e^{-ta^2(\xi)}$ is positive definite. We define for $s \geq 0$ the norm

$$\|u\|^2_{a^2,s} = \int_{\mathbb{R}^N} (1 + a^2(\xi))^{2s} |\hat{u}(\xi)| d\xi$$

and the anisotropic Sobolev spaces

$$H^{a^2,s}(\mathbb{R}^N) = \{u \in L_2(\mathbb{R}^N) : \|u\|_{a^2,s} < \infty\}.$$

The space $H^{a^2,s}(\mathbb{R}^N)$ is a real Hilbert space with the scalar product

$$(u, v)_{a^2,s} = \int_{\mathbb{R}^N} (1 + a^2(\xi))^{2s} \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi$$

and $C_0^\infty(\mathbb{R}^N)$ is a dense subspace of $H^{a^2,s}(\mathbb{R}^N)$. For $a^2(\xi) = |\xi|^2$ the space $H^{a^2,s}(\mathbb{R}^N)$ coincides with the usual Sobolev space $H^{2s}(\mathbb{R}^N)$.

It is known that a real valued continuous negative definite function a^2 satisfies for some $c > 0$ the estimate

$$0 \leq a^2(\xi) \leq c(1 + |\xi|^2). \tag{3}$$

Suppose also that for some $r \in (0, 1]$ the function a^2 satisfies the condition

(a_r) $\quad a^2(\xi) \geq c|\xi|^{2r}$ for some $c > 0$ and all $\xi \in \mathbb{R}^N$, $|\xi| \geq R > 0$.

Then $H^{a^2,s}(\mathbb{R}^N)$ is continuously embedded in $H^{2sr}(\mathbb{R}^N)$. Using embedding theorems for the Sobolev scale $H^t(\mathbb{R}^N)$ we can obtain embedding results for $H^{a^2,s}(\mathbb{R}^N)$.

In the following we will always suppose that $p : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a real–valued continuous symbol such that for any fixed $x \in \mathbb{R}^N$ the function $p(x, \cdot) : \mathbb{R}^N \to \mathbb{R}$ is negative definite and $p(x, \xi)$ has the decomposition

$$p(x, \xi) = p_1(\xi) + p_2(x, \xi)$$

where for a suitable $m \in \mathbb{N}$

(p_1) $|p_1(\xi)| \leq c(1 + a^2(\xi))$ for some $c > 0$ and all $\xi \in \mathbb{R}^N$;

(p_2) $p_2(\cdot, \xi) \in C^m(\mathbb{R}^N)$ and for all $\beta \in \mathbb{N}_0^n$, $|\beta| \leq m$,

$$|\partial_x^\beta p_2(x, \xi)| \leq \varphi_\beta(x)(1 + a^2(\xi))$$

holds for all $\xi \in \mathbb{R}^N$ with some $\varphi_\beta \in L_1(\mathbb{R}^N)$;

(p_3) $p_1(\xi) \geq 2\gamma_0 a^2(\xi)$ for some $\gamma_0 > 0$ and all $\xi \in \mathbb{R}^N$, $|\xi| \geq R > 0$;

(p_4) $\Sigma_{|\alpha| \leq m} \|\varphi_\alpha\|_{L_1}$ is small w.r.t. γ_0 (in a very precise sense, see [7]).

Then the operator $p(x, D)$ as defined in (1) maps $C_0^\infty(\mathbb{R}^N)$ into the space $C(\mathbb{R}^N)$ and the bilinear form associated with $p(x, D)$

$$B(u, v) = \int_{\mathbb{R}^N} p(x, D)u(x) \cdot v(x)dx$$

is defined for $u, v \in C_0^\infty(\mathbb{R}^N)$. In the following we will suppose that the operator $p(x, D)$ is symmetric on $C_0^\infty(\mathbb{R}^N)$. Then $p(x, D)$ has a selfadjoint extension on $L_2(\mathbb{R}^N)$ with domain $H^{a^2,1}(\mathbb{R}^N)$. The bilinear form B extends to a continuous symmetric Dirichlet form with domain $H^{a^2,1/2}(\mathbb{R}^N)$, see [5, 11] for the general theory of Dirichlet forms and their properties. In particular the form B is positive definite on $H^{a^2,1/2}(\mathbb{R}^N)$, i.e.

$$B(u, u) \geq 0 \quad \text{for all} \quad u \in H^{a^2,1/2}(\mathbb{R}^N).$$

Moreover the form B satisfies Gårding inequality

$$B(u, u) \geq \gamma_0 \|u\|^2_{a^2,1/2} - \lambda_0 \|u\|^2_{L_2}, \tag{4}$$

here γ_0 is taken from condition (p_3) and $\lambda_0 > 0$.

3 Variational settings of the problem

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ be an open set with smooth boundary $\partial\Omega$ and $u \in C_0^\infty(\Omega)$. We can extend u to $\mathbb{R}^N$ by setting it in Ω^c equal to zero obtaining a function in $C_0^\infty(\mathbb{R}^N)$ with support in Ω. For this reason we can identify $C_0^\infty(\Omega)$ as a subspace of $C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N) \subseteq H^{a^2,s}(\mathbb{R}^N)$ we can take the closure of $C_0^\infty(\Omega)$ in $H^{a^2,1/2}(\mathbb{R}^N)$ which we denote by $H_0^{a^2,1/2}(\Omega)$. Suppose that $u \in H_0^{a^2,1/2}(\Omega)$. For any $\varphi \in C_0^\infty(\Omega^c)$ we find

$$\int_{\mathbb{R}^N} u(x)\varphi(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} u_n(x)\varphi(x)dx = 0$$

where $(u_n) \subset C_0^\infty(\Omega)$ converges to u in the norm $\|\cdot\|_{a^2,1/2}$. Thus we find $u = 0$ a.e. in Ω^c showing that elements in $H_0^{a^2,1/2}(\Omega)$ fulfill the "boundary" condition in a generalized sense.

It is shown in [7] that under condition (a_r) the space $H_0^{a^2,1/2}(\Omega)$ is continuously embedded into the standard Sobolev space $H_0^r(\Omega)$. Using the Sobolev embedding theorems we obtain the sequence of continuous embedding

$$H_0^{a^2,1/2}(\Omega) \subseteq L_{\frac{2N}{N-2r}}(\Omega) \subseteq L_2(\Omega) \subseteq L_{\frac{2N}{N+2r}}(\mathbb{R}^N). \tag{5}$$

Moreover $L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$ embedded into $[H_0^{a^2,1/2}(\Omega)]^*$ in the sense that

$$l_h(u) = \int_{\mathbb{R}^N} h(x)u(x)dx$$

is a linear continuous functional on $H_0^{a^2,1/2}(\Omega)$ for each $h \in L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$. Since Ω is bounded the embedding of $H_0^{a^2,s}(\Omega)$ into $L_2(\Omega)$ is compact and there exists the precise embedding constant $\sigma = \sigma(\Omega) > 0$ such that

$$\|u\|_{L_2(\mathbb{R}^N)} \leq \sigma \|u\|_{a^2,1/2} \quad \text{for all} \quad u \in H_0^{a^2,1/2}(\Omega). \tag{6}$$

Combining (6) with Gårding's inequality (4) we obtain the estimate

$$B(u,u) \geq (\gamma_0 - \lambda_0\sigma^2)\|u\|^2_{a^2,1/2} \quad \text{for all} \quad u \in H_0^{a^2,1/2}(\Omega), \tag{7}$$

that is we can assert that B is strictly positive definite for domains Ω with sufficiently small embedding constant $\sigma(\Omega)$. However the form B is always positive definite on $H_0^{a^2,1/2}(\Omega)$ since B is Dirichlet form and we can replace (7) by the estimate

$$B(u,u) \geq (0 \vee (\gamma_0 - \lambda_0\sigma^2))\|u\|^2_{a^2,1/2} \quad \text{for all} \quad u \in H_0^{a^2,1/2}(\Omega). \tag{8}$$

Now let $u \in H_0^{a^2,1/2}(\Omega)$ such that $p(x,D)u \in L_2(\mathbb{R}^N)$ be a solution of the Dirichlet problem (2) and suppose that the function $f(x,u(x))$ is integrable. Multiplying with $\varphi \in C_0^\infty(\Omega)$ we find

$$B(u,\varphi) = \int_{\mathbb{R}^N} f(x,u(x))\varphi(x)dx \tag{9}$$

for all $\varphi \in C_0^\infty(\Omega)$. Conversely, it is clear that if $u \in H_0^{a^2,1/2}(\Omega)$ satisfies (9) for all $\varphi \in C_0^\infty(\Omega)$ and $p(x,D)u \in L_2(\mathbb{R}^N)$ then u is a solution of (2). For this reason we will say that u is *a weak solution* of the Dirichlet problem (2) if (9) holds for all $\varphi \in C_0^\infty(\Omega)$.

Let us note that in general the function $f(x,\cdot)$ has a nontrivial dependence on x even on the complement of Ω. For example the class of functions $f(x,u) = g(u) + h(x)$ with $supp\,(h) = \mathbb{R}^N$ is admissible. It is easy to see that actually the solutions of Dirichlet problem (2) do not depend on the behaviour of $x \mapsto f(x,u)$

on Ω^c. It is possible to assume that $f(x,u) \equiv 0$ on Ω^c. However we prefer to consider the more general case $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ because of the probabilistic motivation of the Dirichlet problem (2), see [8] for the discussion in the linear case.

We define *the energy functional* J for problem (2) by means of formula

$$J(u) = \frac{1}{2}B(u,u) - \int_{\mathbb{R}^N} F(x,u(x))dx,$$

here

$$F(x,u) = \int_0^u f(x,\xi)\, d\xi$$

is the primitive of f with respect to the second variable, note that $F(x,0) \equiv 0$.

We are interested in the conditions on a nonlinearity $f(x,u)$ which ensure that the energy functional J is well defined on $H_0^{a^2,1/2}(\Omega)$ and each local minimum of J corresponds to a weak solution of the original boundary value problem (2).

Lemma 1 *Suppose that $f(x,u)$ satisfies the assumption*

(f_r) *there exists $c > 0$ and $h \in L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$ such that*

$$|f(x,u)| \le c|u|^{\frac{N+2r}{N-2r}} + h(x).$$

Then the functional J is defined on $H_0^{a^2,1/2}(\Omega)$.

Proof. From condition (f_r) it follows that the primitive $F(x,u)$ satisfy the estimate

$$|F(x,u)| \le c_1|u|^{\frac{2N}{N-2r}} + c_2h(x)u.$$

Then

$$\int_{\mathbb{R}^N} F(x,u(x))dx \le c_1 \int_\Omega |u(x)|^{\frac{2N}{N-2r}}dx + c_2 \int_{\mathbb{R}^N} h(x)u(x)dx <$$

$$c_1(\|u\|_{L_{\frac{2N}{N-2r}}})^{\frac{2N}{N-2r}} + c_2\|h\|_{L_{\frac{2N}{N+2r}}}\|u\|_{L_{\frac{2N}{N-2r}}} < \infty$$

since $u \in H_0^{a^2,1/2}(\Omega) \subseteq L_{\frac{2N}{N-2r}}(\Omega)$ and $h \in L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$. □

Lemma 2 *Suppose that assumption (f_r) holds. Then the functional J is Gâteaux differentiable on the space $H_0^{a^2,1/2}(\Omega)$ and its derivative for all $\varphi \in H_0^{a^2,1/2}(\Omega)$ is given by the formula*

$$J'(u)(\varphi) = B(u,\varphi) - \int_{\mathbb{R}^N} f(x,u(x))\varphi(x)dx. \tag{10}$$

Moreover each local minimum $u \in H_0^{a^2,1/2}(\Omega)$ of the functional J is a weak solution of the Dirichlet problem (2).

Proof. Clearly $B(u,u)$ is differentiable on $H_0^{a^2,1/2}(\Omega)$ as a continuous bilinear form and its derivative for all $\varphi \in H_0^{a^2,1/2}(\Omega)$ is given by the formula

$$B'(u,u)(\varphi) = B(u,\varphi).$$

We check the differentiability of the nonlinear term

$$J_F(u) = \int_{\mathbb{R}^N} F(x,u(x))dx.$$

By the mean value theorem for each $u \in H_0^{a^2,1/2}(\Omega)$ and $\varphi \in H_0^{a^2,1/2}(\Omega)$ there exists a function $\theta(x)$ such that $0 \le \theta(x) \le 1$ and

$$\frac{J_F(u+\tau\varphi) - J_F(x)}{\tau} = \int_\Omega f(x, u(x) + \tau\theta(x)\varphi(x))\varphi(x)dx. \tag{11}$$

It is known [1] that the function θ may be chosen to be measurable so $\theta \in L_\infty(\Omega)$ and the right hand side of (11) makes sense.

We shall verify that the integral on the right hand side of (11) does exist. Recall that $H_0^{a^2,1/2}(\Omega) \subseteq L_{\frac{2N}{N-2r}}(\Omega)$ by the sequence of embedding (5). Hence we have for $\tau \in \mathbb{R}$

$$u(x) + \tau\theta(x)\varphi(x) \in L_{\frac{2N}{N-2r}}(\Omega)$$

Further from assumption (f_r) it follows that

$$f(x, u(x) + \tau\theta(x)\varphi(x)) \in L_{\frac{2N}{N+2r}}(\mathbb{R}^N),$$

see cf. [1]. Since $\varphi \in H_0^{a^2,1/2}(\Omega) \subseteq L_{\frac{2N}{N-2r}}(\Omega)$ we have

$$f(x, u(x) + \tau\theta(x)\varphi(x))\varphi(x) \in L_1(\mathbb{R}^N).$$

Hence the integral in the right hand side of (11) does exist.

Let $|\tau| \le 1$ and $\tau \to 0$. Clearly

$$u(x) + \tau\theta(x)\varphi(x) \to u(x)$$

in measure and form an U-bounded family of functions in $L_{\frac{2N}{N-2r}}(\Omega)$, i.e. there exists $U \in L_{\frac{2N}{N-2r}}(\Omega)$ such that

$$|u(x) + \tau\theta(x)\varphi(x)| \le U(x) \quad \text{for all} \quad \tau \in [-1,1].$$

Hence for $|\tau| \le 1$ and $\tau \to 0$

$$f(x, u(x) + \tau\theta(x)\varphi(x)) \to f(x, u(x))$$

in measure and makes an U-bounded family of functions in $L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$. So the Lebesgue dominated convergence theorem can be applied to (11) and we have

$$J_F'(u)(\varphi) = \frac{d}{d\tau} J_F(u+\tau\varphi)\,|_{\tau=0} = \lim_{\tau\to 0} \frac{J_F(u+\tau\varphi) - J_F(u)}{\tau} =$$

$$\lim_{\tau\to 0}\int_{\Omega} f(x,u(x)+\tau\theta(x)\varphi(x))\varphi(x)dx = \int_{\Omega} f(x,u(x))\varphi(x)dx.$$

Since $f(x,u(x)) \in L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$ it follows that $J'_F(u,\cdot)$ is a linear continuous functional on $H_0^{a^2,1/2}(\Omega)$ and therefore J_F is Gŝteaux differentiable on $H_0^{a^2,1/2}(\Omega)$.

Finally let $u \in H_0^{a^2,1/2}(\Omega)$ be a minimum for J. Clearly (f_r) implies that the function $f(x,u(x))$ is integrable. By the classical Euler–Fermat principle we have

$$J'(u)(\varphi) = B(u,\varphi) - \int_{\mathbb{R}^N} f(x,u(x))\varphi(x)dx = 0$$

for all $\varphi \in H_0^{a^2,1/2}(\Omega)$. Hence (9) holds for all $\varphi \in C_0^\infty(\Omega)$ and therefore is u is a weak solution for the Dirichlet problem (2). □

Remark 1 By a standard arguments we can also prove that actually under assumption (f_r) the functional J is continuously differentiable on $H_0^{a^2,1/2}(\Omega)$. This follows by the standard arguments from the continuity of the embedding $H_0^{a^2,1/2}(\Omega) \subseteq L_{\frac{2N}{n-2r}}$. However we do not need it in the further consideration.

4 Existence of a minimum

In this section we are interested in investigating the conditions which lead to the existence of a minimum for J. According to the classical Weierstrass principle (see, cf. [13]) it suffices to verify that J is coercive and (sequentially) lower semicontinuous with respect to the weak topology on $H_0^{a^2,1/2}(\Omega)$, that is

$$J(u) \to +\infty \quad \text{as} \quad \|u\|_{a^2,1/2} \to \infty.$$

Let F satisfy the usual one–sided *coercive condition*

(F) there exist $\alpha < 0 \vee (\frac{\gamma_0}{\sigma^2} - \lambda_0)$ and a functions $h \in L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$, $g \in L_1(\mathbb{R}^N)$ such that

$$F(x,u) \le \frac{\alpha}{2}u^2 + h(x)u + g(x),$$

here γ_0, λ_0 is a taken from Gårding inequality (4) and σ is the embedding constant from (6). We assert that under the condition (F) the functional J attains its infimum.

Lemma 3 *Suppose that assumptions $(f_r), (F)$ holds. Then the functional J is bounded from below and coercive.*

Proof. From (F) and estimates (6), (8) it follows that

$$J(u) \ge \frac{1}{2}B(u,u) - \int_{\mathbb{R}^N}\left\{\frac{\alpha}{2}u^2(x) + h(x)u(x) + g(x)\right\}dx =$$

$$\frac{1}{2}\left(B(u,u) - \alpha\|u\|^2_{L_2}\right) - \int_{\mathbb{R}^N} h(x)u(x)dx - \int_{\mathbb{R}^N} g(x)dx \ge$$

$$\frac{1}{2}\left((0 \vee (\gamma_0 - \lambda_0\sigma^2)) - \alpha\sigma^2\right)\|u\|^2_{a^2,1/2} - \|h\|_{a^2,-1/2}\|u\|^2_{a^2,1/2} - \|g\|_{L_1} \to +\infty$$

as $\|u\|_{a^2,1/2} \to +\infty$ since by assumption $\alpha < 0 \vee (\frac{\gamma_0}{\sigma^2} - \lambda_0)$. $\square$

Lemma 4 *Suppose that assumptions $(f_r), (F)$ holds. Then the functional J is lower semi–continuous in the weak topology of $H_0^{a^2,1/2}(\Omega)$.*

Proof. The energy functional J can be rewritten in the form:

$$J(u) = \frac{1}{2}B(u,u) - \int_{\mathbb{R}^N} F(x,u(x))dx \; =$$

$$\frac{1}{2}B(u,u) - \int_{\mathbb{R}^N} \frac{\alpha}{2}u^2(x) + h(x)u(x) + g(x)dx +$$

$$\int_{\mathbb{R}^N} \left\{\frac{\alpha}{2}u^2(x) + h(x)u(x) + g(x)\right\} - F(x,u(x))dx \; =$$

$$\left\{\frac{1}{2}B(u,u) - \frac{\alpha}{2}\|u\|^2_{L_2}\right\} - \int_{\mathbb{R}^N} h(x)u(x)dx - \int_{\mathbb{R}^N} g(x)dx +$$

$$\int_{\mathbb{R}^N} \left\{\frac{\alpha}{2}u^2(x) + h(x)u(x) + g(x)\right\} - F(x,u(x))dx.$$

Let us consider each term separately. We have by (F) and (6),(8)

$$\frac{1}{2}B(u,u) - \frac{\alpha}{2}\|u\|^2_{L_2} \geq \frac{1}{2}(0 \vee (\gamma_0 - \lambda_0\sigma^2) - \alpha\sigma^2)\|u\|^2_{a^2,1/2} \geq 0$$

for all $u \in H_0^{a^2,1/2}(\Omega)$. Hence the quadratic term is (sequentially) weakly lower semicontinuous as a positive definite form on $H_0^{a^2,1/2}(\Omega)$ (see cf. [14]). Also the linear term generated by $h \in L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$ is continuous and hence weakly continuous on $H_0^{a^2,1/2}(\Omega)$.

Now let us consider the last term

$$J_F(u) = \int_{\mathbb{R}^N} \left\{\frac{\alpha}{2}u^2(x) + h(x)u(x) + g(x)\right\} - F(x,u(x))dx.$$

Let $(u_n) \subset H_0^{a^2,1/2}(\Omega)$ be a sequence weakly converging to u_0. Then (u_n) is bounded in $H_0^{a^2,1/2}(\Omega)$. Since the embedding of $H_0^{a^2,1/2}(\Omega)$ into $L_2(\Omega)$ is compact, (u_n) contains a subsequence converging in $L_2(\Omega)$. It is easy to see that weak convergence in $H_0^{a^2,1/2}(\Omega)$ and convergence in $L_2(\Omega)$ are consistent in the sense that if the sequence converges in both topology then limits coincide. We conclude that (u_n) converges to u_0 in $L_2(\Omega)$ and hence converges to u_0 in measure. Then the sequence

$$v_n(x) = \frac{\alpha}{2}u_n^2(x) + h(x)u_n(x) + g(x) - F(x,u_n(x))$$

also converges to

$$v_0(x) = \frac{\alpha}{2}u_0^2(x) + h(x)u_0(x) + g(x) - F(x,u_0(x))$$

in measure. From (F) it follows that the sequence (v_n) is nonnegative. Now applying the Fatou–Lemma we obtain

$$\int_{\mathbb{R}^N} v_0(x)dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} v_n(x)dx,$$

which means that J_F is (sequentially) weakly lower semicontinuous. $\square$

Theorem 1 *Suppose that assumptions $(f_r), (F)$ holds. Then J is bounded from below and has a point of minimum on $H_0^{a^2,1/2}(\Omega)$. Moreover if the primitive $F(x,u)$ is strictly convex in u then the minimum point of J in $H_0^{a^2,1/2}(\Omega)$ is unique.*

The proof of the theorem follows immediately from Lemmas 3 and 4 (see cf. [13, 14]).

Remark 2 The results of this section remains true without any growth assumptions on $|f(x,u)|$ that is without condition (f_r). Of course in this case J may be not well–defined on the whole space $H_0^{a^2,1/2}(\Omega)$. However under one–sided condition (F) the minimum still exists on $Dom(J) \subseteq H_0^{a^2,1/2}(\Omega)$, see [12] for a close consideration.

5 Solvability - a basic result

The basic existence results for the Dirichlet problem (2) follow immediately from Theorem 1 and Lemma 2.

Theorem 2 *Suppose that assumptions $(f_r), (F)$ holds. Then the Dirichlet problem (2) has at least one weak solution in the space $H_0^{a^2,1/2}(\Omega)$. Moreover if the primitive $F(x,u)$ is strictly convex in u then the solution of (2) in $H_0^{a^2,1/2}(\Omega)$ is unique.*

As an example let us consider the following problem

$$\left\{ \begin{array}{rcll} p(x,D)u + u^\rho & = & h(x) & \text{a.e. in } \Omega, \\ u & = & 0 & \text{a.e. in } \Omega^c. \end{array} \right. \tag{12}$$

Corollary 1 *For each $\rho \in (0, \frac{N-2r}{N+2r})$ and $h \in L_{\frac{2N}{N+2r}}(\mathbb{R}^N)$ the problem (12) has a unique weak solution in the space $H_0^{a^2,1/2}(\Omega)$.*

In the previous sections we discussed the Dirichlet problem (2) using the Dirichlet space $H_0^{a^2,1/2}(\Omega)$ generated by $p(x,D)$. Actually we did not further use the Dirichlet space structure. In the next sections we need this Dirichlet structure in order to develop the truncation technique for problem (2). This technique should allow to avoid the growth restriction (f_r) on the nonlinearity $f(x,u)$ and to obtain some additional information on the properties of the solutions of (2).

6 Truncation of the nonlinearity

In this section we will always suppose that the form B is strictly positive definite, that is $\gamma_0 - \lambda_0\sigma^2 > 0$.

Let us consider the problem

$$\left\{ \begin{array}{rcll} p(x,D)u &=& f(u)+h(x) & \text{a.e. in } \Omega, \\ u &=& 0 & \text{a.e. in } \Omega^c, \end{array} \right. \tag{13}$$

here $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Theorem 3 *Suppose that the function $f(u)$ satisfies assumption*

$$\inf_{u\geq 0} f(u) = -\infty, \qquad \sup_{u\leq 0} f(u) = +\infty. \tag{14}$$

Then for each $h \in L_\infty(\mathbb{R}^N)$ the problem (13) has at least one weak solution $\tilde{u} \in H_0^{a^2,1/2}(\Omega) \cap L_\infty(\Omega)$. Moreover if the function h is nonnegative on $\mathbb{R}^N$ then the solution $\tilde{u}$ is nonnegative on Ω.

Proof. We will follow the lines of the proof in [3, Proposition 1] where the semilinear Dirichlet problem for the Laplacian $-\Delta$ was considered. By (14) there exist constants $a \leq 0 \leq b$ such that

$$f(a) \geq \max\{\sup_{\mathbb{R}^N} h, 0\} \quad \text{and} \quad f(b) \leq \min\{\inf_{\mathbb{R}^N} h, 0\}.$$

We define a truncation of the nonlinearity $f(u)$ by means of formula

$$\tilde{f}(u) = \left\{ \begin{array}{lll} f(a), & \text{if} & u < a, \\ f(u), & \text{if} & a \leq u \leq b, \\ f(b), & \text{if} & u > b. \end{array} \right.$$

The truncation $\tilde{f}$ is a bounded continuous function. Hence $\tilde{f}$ satisfies the growth condition (f_r) and the primitive

$$\tilde{F}(u) = \int_0^u \tilde{f}(\xi)d\xi$$

satisfies the coercivity condition (F) since $\gamma_0 - \lambda_0\sigma^2 > 0$. Therefore by Theorem 2 the truncated problem

$$\left\{ \begin{array}{rcll} p(x,D)u &=& \tilde{f}(u)+h(x) & \text{a.e. in } \Omega, \\ u &=& 0 & \text{a.e. in } \Omega^c. \end{array} \right.$$

has at least one weak solution $\tilde{u} \in H_0^{a^2,1/2}(\Omega)$.

We will prove that

$$a \leq \tilde{u}(x) \leq b \quad \text{a.e. in } \Omega.$$

Then $\tilde{u}$ is a (bounded) solution of original problem (13).

Let $(\tilde{u}-b)^+ \geq 0$ be a test function. Since $H_0^{a^2,1/2}(\Omega)$ is a Dirichlet space $(\tilde{u}-b)^+ \in H_0^{a^2,1/2}(\Omega)$. Then we have

$$B(\tilde{u},(\tilde{u}-b)^+) = \int_{\mathbb{R}^N} (\tilde{f}(\tilde{u}(x)) + h(x))(\tilde{u}-b)^+(x)dx =$$

$$\int_{supp\,((\tilde{u}-b)^+)} (f(\tilde{u}(x)) + h(x))(\tilde{u}-b)^+(x)dx \leq 0 \tag{15}$$

since

$$f(\tilde{u}(x)) + h(x) = f(b) + h(x) \leq 0 \quad \text{on} \quad supp\,((\tilde{u}-b)^+)$$

by the definition of $\tilde{f}$. Further

$$0 \leq B((\tilde{u}-b)^+,(\tilde{u}-b)^+) = B(\tilde{u},(\tilde{u}-b)^+) - B(\tilde{u}\wedge b,(\tilde{u}-b)^+) \leq 0$$

by (15) and since for all $b \geq 0$

$$B(\tilde{u}\wedge b,(\tilde{u}-b)^+) \geq 0$$

by the property of Dirichlet forms (see e.g. [11, p.32]). Therefore $(\tilde{u}-b)^+ = 0$ and $\tilde{u} \leq b$. In the same way taking as a test function $(\tilde{u}+a)^- \geq 0$ we can show that $\tilde{u} \geq a$. In particular if the function h is nonnegative we have $a = 0$. This means that $\tilde{u} \geq 0$. □

Corollary 2 *For each $\rho > 0$ and $h \in L_\infty(\mathbb{R}^N)$ the problem (12) has at least one weak solution $\tilde{u} \in H_0^{a^2,1/2}(\Omega) \cap L_\infty(\Omega)$. Moreover if the function h is nonnegative (nonpositive) on $\mathbb{R}^N$ then the solution $\tilde{u}$ is nonnegative (nonpositive) on Ω.*

Now, consider the problem

$$\begin{cases} p(x,D)u = \lambda f(u) & \text{a.e. in } \Omega, \\ u = 0 & \text{a.e. in } \Omega^c, \end{cases} \tag{16}$$

here $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(0) = 0$ and $\lambda > 0$ is a real parameter. Clearly $u = 0$ is a trivial solution of (16). We are interesting in the existence of nontrivial solutions.

Theorem 4 *Suppose that the function $f(u)$ satisfies the assumption*

$$\liminf_{u\to+\infty} f(u) < 0, \tag{17}$$

and there exists $\xi > 0$ such that $f(\xi) > 0$. Then for each $\lambda > 0$ sufficiently large the problem (16) has at least one nontrivial nonnegative weak solution $\tilde{u} \in H_0^{a^2,1/2}(\Omega) \cap L_\infty(\Omega)$.

Proof. The arguments in the case of the Dirichlet problem for the Laplacian $-\Delta$ is well–known, see cf. [2]. By (17) and the continuity of f there exist constants $b > \xi$ such that $f(b) = 0$. We define a truncation of the nonlinearity $f(u)$ by means of formula

$$\tilde{f}(u) = \begin{cases} 0, & \text{if} \quad u < 0, \\ f(u), & \text{if} \quad 0 \leq u \leq b, \\ 0, & \text{if} \quad u > b. \end{cases}$$

The truncation $\tilde{f}$ is a bounded continuous function. Let us note that for all $\lambda > 0$ the nonlinearity $\lambda \tilde{f}$ has the same "zeros" as $\tilde{f}$. Therefore $\lambda \tilde{f}$ satisfies the growth condition (f_r), the primitive $\lambda \tilde{F}$ satisfies the coercivity condition (F), and by Theorem 1 the truncated functional

$$\tilde{J}_\lambda(u) = \frac{1}{2} B(u,u) - \lambda \int_\Omega \tilde{F}(u(x))dx$$

has a point of minimum $\tilde{u}_\lambda \in H_0^{a^2,1/2}(\Omega)$ for each $\lambda > 0$. Clearly $\tilde{u}_\lambda$ is a weak solution of the "truncated" problem

$$\begin{cases} p(x,D)u &= \lambda \tilde{f}(u) \qquad \text{a.e. in } \Omega, \\ u &= 0 \qquad \text{a.e. in } \Omega^c. \end{cases}$$

We will prove that

$$0 \leq \tilde{u}_\lambda(x) \leq b \quad \text{a.e. in } \Omega.$$

Then $\tilde{u}_\lambda$ is a (bounded) solution of original problem (16).

Let $(\tilde{u}_\lambda - b)^+ \in H_0^{a^2,1/2}(\Omega)$ be a test function. Then we have

$$B(\tilde{u}_\lambda, (\tilde{u}_\lambda - b)^+) = \lambda \int_{\mathbb{R}^N} \tilde{f}(\tilde{u}_\lambda(x))(\tilde{u}_\lambda - b)^+(x)dx =$$

$$\lambda \int_{supp\,((\tilde{u}_\lambda - b)^+)} \tilde{f}(\tilde{u}_\lambda(x))(\tilde{u}_\lambda - b)^+(x)dx \leq 0 \tag{18}$$

since

$$\tilde{f}(\tilde{u}_\lambda(x)) = f(b) = 0 \quad \text{on} \quad supp\,((\tilde{u}_\lambda - b)^+)$$

by the definition of $\tilde{f}$. Further

$$0 \leq B((\tilde{u}_\lambda - b)^+, (\tilde{u}_\lambda - b)^+) = B(\tilde{u}_\lambda, (\tilde{u}_\lambda - b)^+) - B(\tilde{u}_\lambda \wedge b, (\tilde{u}_\lambda - b)^+) = 0$$

by (18) and since $B(\tilde{u}_\lambda \wedge b, (\tilde{u}_\lambda - b)^+) \geq 0$ for $b \geq 0$ by the property of Dirichlet forms. Therefore $(\tilde{u}_\lambda - b)^+ = 0$ and $\tilde{u}_\lambda \leq b$.

Similarly taking as a test function $(\tilde{u}_\lambda)^- \in H_0^{a^2,1/2}(\Omega)$ we obtain

$$B(\tilde{u}_\lambda, (\tilde{u}_\lambda)^-) = \lambda \int_{\mathbb{R}^N} \tilde{f}(\tilde{u}_\lambda(x))(\tilde{u}_\lambda)^-(x)dx = 0 \tag{19}$$

and

$$0 \leq B((\tilde{u}_\lambda)^-, (\tilde{u}_\lambda)^-) = B((\tilde{u}_\lambda)^+, (\tilde{u}_\lambda)^-) - B(\tilde{u}_\lambda, (\tilde{u}_\lambda)^-) \leq 0$$

by (19) and since

$$B((\tilde{u}_\lambda)^+, (\tilde{u}_\lambda)^-) \leq 0$$

by the property of Dirichlet forms (see e.g. [11, p.33]). Hence $(\tilde{u}_\lambda)^- = 0$ and $\tilde{u}_\lambda \geq 0$.

Finally, we will show that $\tilde{u}_\lambda \neq 0$ for $\lambda > 0$ sufficiently large. Let us note that by condition (f_r) the functional

$$J_F(u) = \int_{\mathbb{R}^N} F(u(x))dx$$

is defined and continuous on the space $L_{\frac{2N}{N-2r}}(\Omega)$, see the proof of Lemma 1 and [1]. Let $u_0(x) \equiv \xi$ on Ω. Clearly $u_0 \in L_{\frac{2N}{N-2r}}(\Omega)$ and $J_F(u_0) > 0$. Further the space $H_0^{a^2,1/2}(\Omega)$ is densely embedded into $L_{\frac{2N}{N-2r}}(\Omega)$. By continuity arguments for ε small enough we can take an element $u_\varepsilon \in H_0^{a^2,1/2}(\Omega)$ such that

$$J_F(u_\varepsilon) \geq J_F(u_0) - \varepsilon > 0.$$

Then for $u_\varepsilon \in H_0^{a^2,1/2}(\Omega)$ we obtain

$$J_\lambda(u_\varepsilon) = \frac{1}{2}B(u_\varepsilon, u_\varepsilon) - \lambda J_F(u_\varepsilon) < \frac{1}{2}B(u_\varepsilon, u_\varepsilon) - \lambda(J_F(u_0) - \varepsilon) < 0$$

for $\lambda > 0$ sufficiently large. Hence

$$\min_{H_0^{a^2,1/2}(\Omega)} J_\lambda < 0.$$

Since $J_\lambda(0) = 0$ by definition of the primitive $\tilde{F}$ and $\tilde{u}_\lambda$ is a minimum of J_λ it follows that $\tilde{u}_\lambda \neq 0$ for all $\lambda > 0$ sufficiently large. □

Remark 3 In this section we considered just simple examples showing that the typical truncation technique known for the second–order elliptic partial differential operators can be applied to the nonlocal Dirichlet problem (2). It seems that much more delicate and involved results could be obtained by a combination of sub– and super–solution techniques with topological methods of critical points theory, see cf. [13] for the case of local problems.

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