Positive solutions to sublinear second–order divergence type elliptic equations in cone–like domains

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Abstract

We study the existence and nonexistence of positive solutions to a sublinear ($p < 1$) second–order divergence type elliptic equation $(*): \ -\nabla\cdot a\cdot\nabla u = u^p$ in unbounded cone–like domains $\mathcal{C}_\Omega$. We prove the existence of the critical exponent

$$p_*(a, \mathcal{C}_\Omega) = \sup\{p < 1 : \ (*) \textit{ has a positive supersolution at infinity in } \mathcal{C}_\Omega \},$$

which depends on the geometry of the cone $\mathcal{C}_\Omega$ and the coefficients a of the equation.

1 Introduction

We study the existence and nonexistence of positive (super) solutions to a sublinear second–order divergence type elliptic equation

$$(1.1) \qquad -\nabla\cdot a\cdot\nabla u = u^p \quad \text{in } \ \mathcal{C}_\Omega.$$

Here $p < 1$ is a sublinear (possibly negative) exponent, $\mathcal{C}_\Omega$ is a cone–like domain in $\mathbb{R}^N$ ($N \geq 2$) defined as

$$\mathcal{C}_\Omega := \{(r, \omega) \in \mathbb{R}^N : \ \omega \in \Omega, \ r > 0\},$$

where (r, ω) are the polar coordinates in $\mathbb{R}^N$, cross–section $\Omega \subseteq S^{N-1}$ is a subdomain (a connected open subset) of the unit sphere S^{N-1} in $\mathbb{R}^N$, and

$$-\nabla\cdot a\cdot\nabla := -\sum \partial_{x_i}\left(a_{ij}(x)\,\partial_{x_j}\right)$$

is the second order divergence type elliptic expression generated by a real symmetric measurable and uniformly elliptic matrix $a = (a_{ij}(x))$ on $\mathbb{R}^N$, so that

$$(1.2) \qquad \nu|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and almost all } x \in \mathbb{R}^N,$$

with an ellipticity constant $\nu = \nu(a) > 0$.

Solutions and super-solutions to equation (1.1) are understood in the weak sense. More precisely, we say that u is a *(super) solution* to (1.1) in an open domain $G \subseteq \mathcal{C}_\Omega$ if $u \in H^1_{loc}(G)$ and

$$\int_G \nabla u\cdot a\cdot\nabla\varphi\,dx\,(\geq) = \int_G u^p\varphi\,dx \quad \text{for all } \ 0 \leq \varphi \in H^1_c(G),$$

where $H^1_c(G)$ stands for the set of compactly supported elements from $H^1_{loc}(G)$. By the weak Harnack inequality, any nontrivial nonnegative supersolution to (1.1) in G is strictly positive in G, that is $u^{-1} \in L^\infty_{loc}(G)$. In particular, positive solution are well defined for negative values of the exponent p.

We say that equation (1.1) has a *(super) solution at infinity in* $\mathcal{C}_\Omega$ if there exists a closed ball $\bar{B}_\rho$ centered at the origin with radius $\rho > 1$ such that (1.1) has a (super) solution in $\mathcal{C}_\Omega \setminus \bar{B}_\rho$. We define the *critical exponent* to equation (1.1) by

$$p_* = p_*(a, \mathcal{C}_\Omega) = \sup\{p < 1 : (1.1) \text{ has a positive supersolution at infinity in } \mathcal{C}_\Omega\}.$$

If no positive supersolutions at in infinity in $\mathcal{C}_\Omega$ exists for any $p < 1$ then $p_*(a, \mathcal{C}_\Omega) = -\infty$.

"Critical exponent" type results for equations (1.1) with $p > 1$ have a long history, cf. [8] for a survey of classical and recent work in the area. Equations (1.1) with $p < 1$ are less studied. It is well–known that $p_*(a, \mathcal{C}_{S^{N-1}}) = -\infty$, see [2, 7]. Recently it was established in [6] that in the case of the Laplace operator ($a = id$) equation (1.1) admits a finite critical exponent on proper conical domains. Precisely, in [6] it was proved that $p_*(id, \mathcal{C}_\Omega) = 1 - 2/\alpha_+$, where α_+ is the largest root of the equation $\alpha(\alpha + N - 2) = \lambda_1(\Omega)$ and $\lambda_1(\Omega)$ is the principal Dirichlet eigenvalue of the Laplace–Beltrami operator on Ω. In this paper we investigate properties of the critical exponent $p_*(a, \mathcal{C}_\Omega)$ in the case of general divergence type elliptic equations on cone–like domains. The following proposition collects some elementary properties of the critical exponent.

Proposition 1.1. *Let* $\Omega' \subset \Omega \subseteq S^{N-1}$ *are subdomains of* S^{N-1}*. Then*

(*i*) $-\infty \le p_*(a, \mathcal{C}_\Omega) \le p_*(a, \mathcal{C}_{\Omega'}) \le 1$;

(*ii*) *Equation* (1.1) *admits a positive solution at infinity in* $\mathcal{C}_\Omega$ *for every* $p < p_*(a, \mathcal{C}_\Omega)$.

Remark 1.2. Assertion (i) follows directly from the definition of the critical exponent $p_*(a, \mathcal{C}_\Omega)$ and the fact that $p_*(a, \mathcal{C}_{S^{N-1}}) = -\infty$. Property (ii) simply means that the critical exponent $p_*(a, \mathcal{C}_\Omega)$ divides the semiaxes $[-\infty, 1]$ into precisely one existence and one nonexistence region. Moreover, the existence of a positive supersolution at infinity implies the existence of a positive solution at infinity. The proof of (ii) is similar to the proof of [5, Proposition 1.1]. We omit the details.

We say that Ω is a *proper* subdomain of S^{N-1} and write $\Omega \Subset S^{N-1}$, if $S^{N-1} \setminus \Omega$ contains an open set. The main result of the paper says that similarly to the Laplace equations, divergence type equations on proper cone–like domain admit a nontrivial critical exponent.

Theorem 1.3. *Let* $\Omega \Subset S^{N-1}$ *be a proper subdomain. Then for any uniformly elliptic matrix* a *one has* $p_*(a, \mathcal{C}_\Omega) \in (-\infty, 1)$.

The value of the critical exponent essentially depends on the matrix a and can not be explicitly controlled without further restrictions on the properties of a.

Theorem 1.4. *Let* $\Omega \Subset S^{N-1}$ *be a proper subdomain. Then for any* $p \in (-\infty, 1)$ *there exists a uniformly elliptic matrix* a_p *such that* $p_*(a_p, \mathcal{C}_\Omega) = p$.

Remark 1.5. Theorems 1.3 and 1.4 were announced in [8]. Related results for *superlinear* equations ($p > 1$) of type (1.1) were established in the article [5]. In many aspects the current paper can be seen as a continuation of [5].

In the remaining part of the paper we prove Theorems 1.3 and 1.4. In Section 2 we collect preliminary results concerning associated to (1.1) linear equations. Sections 3 and 4 deal with nonexistence and existence parts of the proof of Theorem 1.3. Section 5 contains the proof of Theorem 1.4.

2 Preliminaries

Let $G \subseteq \mathcal{C}_\Omega$ be an open domain. Consider the linear equation

$$(-\nabla\cdot a\cdot\nabla - V)u = f \quad \text{in } G, \tag{2.1}$$

where $f \in H^{-1}_{loc}(G)$ and $0 \le V \in L^1_{loc}(G)$ is a *form–bounded potential*, that is

$$\int_G Vu^2\,dx \le (1-\epsilon)\int_G \nabla u\cdot a\cdot\nabla u\,dx \quad \text{for all } 0 \le u \in H^1_c(G) \tag{2.2}$$

with some $\epsilon \in (0,1)$. A (super) solution to (2.1) is a function $u \in H^1_{loc}(G)$ such that

$$\int_G \nabla u\cdot a\cdot\nabla\varphi\,dx - \int_G Vu\varphi\,dx\,(\ge) = \langle f,\varphi\rangle \quad \text{for all } 0 \le \varphi \in H^1_c(G),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $H^{-1}_{loc}(G)$ and $H^1_c(G)$. If $u \ge 0$ is a supersolution to

$$(-\nabla\cdot a\cdot\nabla - V)u = 0 \quad \text{in } G, \tag{2.3}$$

then u is a supersolution to $-\nabla\cdot a\cdot\nabla u = 0$ in G, and therefore u satisfies the weak Harnack inequality on any subdomain $G' \Subset G$ (see, e.g. [3, Theorem 8.18]). In particular, every nontrivial supersolution $u \ge 0$ to (2.3) is strictly positive, in the sense that $u^{-1} \in L^\infty_{loc}(G)$.

We define the Hilbert space $D^1_0(G)$ as a completion of $C^\infty_c(G)$ with respect to the norm $\|\nabla u\|_{L_2}$. By the Sobolev inequality, $D^1_0(G) \subset L^{\frac{2N}{N-2}}(G)$. Since the matrix a is uniformly elliptic and the potential $V \ge 0$ is form bounded, the Dirichlet form

$$\mathcal{E}_V(u,v) = \int_G \nabla u\cdot a\cdot\nabla v\,dx - \int_G Vuv\,dx$$

defines an inner product on $D^1_0(G)$. By $D^1(G)$ we denotes the space $D^1(G) = \{u \in L^2_{loc}(G) : \nabla u \in L^2(G)\}$. The next lemma is a standard consequence of the Lax–Milgram Theorem.

Lemma 2.1. *Let $g \in D^1(G)$. Then the problem*

$$(-\nabla\cdot a\cdot\nabla - V)v = 0, \qquad v - g \in D^1_0(G),$$

has a unique solution.

The following two lemmas provide Maximum and Comparison Principles for linear equation (2.3), in a form suitable for our framework (see [5, Lemma 2.2 and Lemma 2.3] for the proofs).

Lemma 2.2. *Let $u \in H^1_{loc}(G)$ be a supersolution to (2.3) such that $u^- \in D^1_0(G)$. Then $v \ge 0$ in G.*

Lemma 2.3. *Let $0 \le u \in H^1_{loc}(G)$ and $v \in D^1_0(G)$. Suppose $u - v$ is a supersolution to (2.3). Then $u \ge v$ in G.*

Here and thereafter, for $0 \le \rho < R \le +\infty$, we denote

$$\mathcal{C}^{(\rho,R)}_\Omega := \{(r,\omega) \in \mathbb{R}^N : \omega \in \Omega,\ r \in (\rho,R)\}.$$

We also use the notation $\mathcal{C}_\Omega^\rho := \mathcal{C}_\Omega^{(\rho,+\infty)}$, so that $\mathcal{C}_\Omega = \mathcal{C}_\Omega^0 = \mathcal{C}_\Omega^{(0,+\infty)}$. Given a function $0 < u \in H^1_{loc}(\mathcal{C}_\Omega^{R/2,R})$ and a subdomain $\Omega' \subseteq \Omega$, denote

$$m_u(R,\Omega') := \inf_{\mathcal{C}_{\Omega'}^{(R/2,R)}} u, \qquad M_u(R,\Omega') := \sup_{\mathcal{C}_{\Omega'}^{(R/2,R)}} u.$$

We also use the standard notation $⨍_G u\,dx := |G|^{-1}\int_G u dx$, with $|G|$ being the Lebesgue measure of a domain $G \subset \mathbb{R}^N$.

An important property of positive supersolutions to homogeneous linear equations in cone–like domains is the following two–sided polynomial bound.

Lemma 2.4. *For any proper subdomain $\Omega' \Subset \Omega$ there exists $\alpha < 2 - N$ and $\beta > 0$ such that the infimum of every supersolution $w > 0$ to the linear equation*

$$-\nabla\cdot a\cdot\nabla w = 0 \quad \text{in } \mathcal{C}_\Omega^\rho. \tag{2.4}$$

satisfies the bound

$$cR^\alpha \le m_w(R,\Omega') \le CR^\beta \qquad (R \gg \rho). \tag{2.5}$$

Proof. We scetch the proof of the upper bound. The derivation of the lower bound is similar.

Let $R > r > \rho$. By the weak Harnack inequality (see, e.g. [3, Theorem 8.18]), w satisfies

$$\inf_{\mathcal{C}_{\Omega'}^{(r,R)}} w \ge C_W ⨍_{\mathcal{C}_{\Omega'}^{(r,R)}} w\,dx,$$

with the weak Harnack constant $C_W \in (0,1)$ which depends on Ω' but not on r and R, as a simple scaling argument shows. Denote $\mu(r) := \inf_{\mathcal{C}_{\Omega'}^{(ra,\,rb)}} w$ and set $a = 1/2$, $b = 3/2$. Let $r > 2\rho$. Then

$$\mu(2r) \;\le\; ⨍_{\mathcal{C}_{\Omega'}^{(2ra,\,2rb)}} w\,dx \le \lambda ⨍_{\mathcal{C}_{\Omega'}^{(ra,\,2rb)}} w\,dx \le \lambda C_W^{-1} \inf_{\mathcal{C}_{\Omega'}^{(ra,2rb)}} w \le C_*\mu(r), \tag{2.6}$$

with $\lambda = \frac{2b-a}{2b-2a}$ and $C_* = \lambda C_W^{-1} > 1$. Let $r_0 > 2\rho$, $r_n = 2^n r_0$ and $n \in \mathbb{N}$. Iterating (2.6) n–times, we obtain

$$\mu(r_n) \le C_*^n \mu(r_0).$$

Choosing $n \in \mathbb{N}$ such that $R < 2ar_n$ and applying once more the weak Harnack inequality, we obtain upper bound (2.5) with $\beta = \log_2 C_*$. □

Remark 2.5. Similar arguments were used by Pinchover [10, Lemma 6.5], compare also [5, Lemma 5.1]. Note that the above proof does not allow to control the value of β and α in terms of the ellipticity constant $\nu(a)$. Note also that on the proper cone–like domains (and in contrast to exterior domains of $\mathbb{R}^N$) the sharp values of β and α in (2.5) essentially depend on the matrix a. For instance, for a given proper cone $\mathcal{C}_\Omega$ and for an arbitrary $\beta > 0$ one can construct a uniformly elliptic matrix a such that equation (2.4) admits a solution $w > 0$ which satisfies $m_w(R,\Omega') \simeq M_w(R,\Omega') \simeq R^\beta$ for all large $R > \rho$, see the operator L_δ, constructed in the proof of Theorem 1.4 below.

3 Proof of Theorem 1.3 – Nonexistence

We begin the proof of nonexistence with the following standard lower bound on positive supersolutions to nonlinear equation (1.1).

Lemma 3.1. *Let $p < 1$ and $u > 0$ be a supersolution at infinity to* (1.1)*. Then for any proper subdomain $\Omega' \Subset \Omega$ there exists $c = c(\Omega')$ such that*

$$m_u(R, \Omega') \geq cR^{\frac{2}{1-p}} \qquad (R \gg 1). \tag{3.1}$$

Proof. Let $w > 0$ be a super-solution at infinity to (1.1). Then $-\nabla \cdot a \cdot \nabla w \geq 0$ in $\mathcal{C}_\Omega^\rho$ for some $\rho \gg 1$ and, by the weak Harnack inequality (see, e.g. [3, Theorem 8.18]), for any $s > 0$ and for any compact $\Omega' \subset \Omega$ $K \subset \mathcal{C}_\Omega^R$ there exists $C_W > 0$ such that

$$\sup_{\mathcal{C}_{\Omega'}^{(R/2,R)}} w^{-1} \leq C_W \left(\fint_{\mathcal{C}_{\Omega'}^{(R/2,R)}} w^{-s} dx \right)^{1/s}. \tag{3.2}$$

The weak Harnack constant $C_W > 0$ depends on Ω' but does not depend on R, as one can see by a simple scaling argument. Further, $w > 0$ is a supersolution to the linearized equation

$$-\nabla \cdot a \cdot \nabla w - \left(w^{p-1}\right) w \geq 0 \quad \text{in } \mathcal{C}_\Omega^\rho,$$

and then it follows from [1, Theorem 3.1] that

$$\int_{\mathcal{C}_\Omega^\rho} \nabla \varphi \cdot a \cdot \nabla \varphi \, dx - \int_{\mathcal{C}_\Omega^\rho} w^{p-1} \varphi^2 \, dx \geq 0 \quad \text{for all } \varphi \in H_c^1(\mathcal{C}_\Omega^\rho) \cap H_c^\infty(\mathcal{C}_\Omega^\rho). \tag{3.3}$$

Fix a proper subdomain $\Omega' \Subset \Omega$. Choose $\psi \in C_c^\infty(\Omega)$ such that $\psi = 1$ on Ω'. For $R \gg \rho$, choose $\theta_R(r) \in C_c^{0,1}(\rho, +\infty)$ such that $0 \leq \theta_R \leq 1$, $\theta_R = 1$ for $r \in [R/2, R]$, *supp* $(\theta_R) = [R/4, 2R]$ and $|\nabla \theta_R| < c/R$. Then

$$\int_{\mathcal{C}_\Omega^\rho} \nabla(\theta_R \psi) \cdot a \cdot \nabla(\theta_R \psi) \, dx \leq cR^{N-2}. \tag{3.4}$$

On the other hand,

$$\int_{\mathcal{C}_{\Omega'}^\rho} w^{p-1} \, (\theta_R \psi)^2 \, dx \geq \int_{\mathcal{C}_{\Omega'}^{(R/2,R)}} w^{p-1} dx. \tag{3.5}$$

Combining (3.3), (3.4) and (3.5) we derive

$$cR^{-2} \geq R^{-N} \int_{\mathcal{C}_{\Omega'}^{(R/2,R)}} w^{p-1} dx = c_0 \fint_{\mathcal{C}_{\Omega'}^{(R/2,R)}} w^{p-1} dx,$$

for some $c_0 > 0$ which does not depend on R. Then by (3.2) with $s = 1 - p$ we obtain

$$cR^{-\frac{2}{1-p}} \geq \left(c_0 \fint_{\mathcal{C}_{\Omega'}^{(R/2,R)}} w^{-(1-p)} \, dx \right)^{\frac{1}{1-p}} \geq c_0^{\frac{1}{1-p}} C_W^{-1} \sup_{\mathcal{C}_{\Omega'}^{(R/2,R)}} w^{-1}.$$

Hence the assertion follows. ∎

Lemma 3.1 combined with the polynomial upper bound from (2.5) on positive supersolutions to the linear equation (2.4) immediately implies an upper bound on the critical exponent $p_*(a, \mathcal{C}_\Omega)$.

Proposition 3.2. *$p_*(a, \mathcal{C}_\Omega) \leq 1 - 2/\beta$, where $\beta > 0$ is taken from* (2.5).

Proof. Fix $p > 1 - 2/\beta$. Assume $u > 0$ is a positive supersolution at infinity to (1.1). Hence u is a positive supersolution to (2.4). But then lower bound (3.1) is incompatible with upper bound (2.5), a contradiction. □

4 Proof of Theorem 1.3 – Existence

To establish a lower bound on the critical exponent $p_*(a, \mathcal{C}_\Omega)$, we consider the linear equation

$$-\nabla \cdot a \cdot \nabla w - V_\epsilon w = 0 \quad \text{in} \ \mathcal{C}_\Omega^\rho, \tag{4.1}$$

where $\rho > 1$,

$$V_\epsilon(x) := \frac{\epsilon}{|x|^2 \log^2 |x|} \wedge 1, \tag{4.2}$$

and $\epsilon > 0$ will be specified later. We are going to show that equation (4.1) on proper cone–like domains always admits a positive (super) solution $w > 0$ that satisfies a lower bound

$$w \geq c|x|^\gamma \quad \text{in} \ \mathcal{C}_\Omega^\rho, \tag{4.3}$$

with some $\gamma > 0$. We call such supersoultion a *growing supersolution* to (4.1).

The construction of a growing (super) solution to (4.1) will be done in several steps. First, we recall the concept of a Green bounded potential, see [4, 7]. Consider the equations

$$-\nabla \cdot a \cdot \nabla v - Vv = 0 \quad \text{in} \quad \mathbb{R}^N, \tag{4.4}$$

where $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$. We say that the potential V is Green bounded if

$$\|V\|_{GB,a} := \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma_a(x, y) V(y) dy < \infty,$$

where $\Gamma_a(x, y)$ is the minimal positive Green function to

$$-\nabla \cdot a \cdot \nabla v = 0 \quad \text{in} \ \mathbb{R}^N.$$

In this case we write $V \in GB$. Note that every Green bounded potential is form bounded in the sense of (2.2) (e.g., by the Stein interpolation theorem).

Note that the condition $\|V\|_{GB,a} < \infty$, is equivalent up to a constant factor to the condition

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{2-N} |V(y)| dy < \infty.$$

In particular, this condition can be used to verify that the potential V_ϵ is Green bounded for sufficiently small $\epsilon > 0$. In what follows we assume that $\epsilon > 0$ is chosen so that $V_\epsilon \in GB$.

We will use the following important property of Green bounded potentials, which was proved in [4], see also further references therein.

Lemma 4.1. *Let $V \in GB$ and $\|V\|_{GB,a} < 1$. Then there exists a quasiconstant solution $w_0 > 0$ to the equation*

$$-\nabla\cdot a\cdot\nabla w - Vw = 0 \quad in \;\; \mathbb{R}^N, \tag{4.5}$$

such that $0 < \epsilon < w_0 < \epsilon^{-1}$ in $\mathbb{R}^N$.

To construct a growing solution to (4.1) we first define a family of approximate solutions. Fix smooth subdomains $U'' \Subset U' \Subset U$ such that $\bar{\Omega} \subset U''$, and a function $0 \lneq \psi \in C_0^\infty(U')$ such that $0 \le \psi \le 1$ and $\psi \equiv 1$ on U''. Let $\theta \in C^\infty[1/2, 1]$ be such that $\theta(1) = 1$, $0 \le \theta \le 1$ and $\theta(1/2) = 0$. Assume that $R \ge 1$ and set $\theta_R(r) := \theta(r/R)$ $(r \in [R/2, R])$. Thus $\theta_R\psi \in D^1(\mathcal{C}_{U'}^{(R/2,R)})$. By $w_{\psi,R}$ we denote the unique solution to the problem

$$-\nabla\cdot a\cdot\nabla w - Vw = 0, \qquad w - \theta_R\psi \in D_0^1(\mathcal{C}_U^{(0,R)}).$$

Observe that $w_{\psi,R}$ depends only on R and ψ, but does not depend on the choice of θ (this easily follows, e.g., from Lemma 2.2). Note also that $w_{\psi,R}$ is positive. Indeed,

$$(w_{\psi,R})^- \le (w_{\psi,R} - \theta_R\psi)^- \in D_0^1(\mathcal{C}_U^{(0,R)}).$$

Thus $w_{\psi,R} > 0$ in $\mathcal{C}_U^{(0,R)}$, by Lemma 2.2 and weak Harnack's inequality.

Lemma 4.2. *There exists $M_\infty > 0$ such that $\|w_{\psi,R}\|_{L^\infty} \le M_\infty$.*

Proof. Let $w_0 > 0$ be a quasiconstant solution to (4.5) that satisfies $0 < \epsilon < w_0 < \epsilon^{-1}$ in $\mathbb{R}^N$. Without loss of generality we may assume that $w_0 \ge \max_U \psi = 1$. Then

$$(-\nabla\cdot a\cdot\nabla - V)((w_0 - \theta_R\psi) - (w_{\psi,R} - \theta_R\psi)) = (-\nabla\cdot a\cdot\nabla - V)(w_0 - w_{\psi,R}) = 0 \quad \text{in } \; \mathcal{C}_U^{(0,R)}.$$

Thus Lemma 2.3 implies that $w_{\psi,R} \le w_0$ in $\mathcal{C}_U^{(0,R)}$, uniformly in $R \ge 1$. □

Fix a compact $K_0 \subset \mathcal{C}_U^{(0,1/2)}$. Set

$$v_{\psi,R} := \frac{w_{\psi,R}}{\inf_{K_0} w_{\psi,R}}.$$

Then $\inf_{K_0} v_{\psi,R} = 1$ and $(v_{\psi,R})_{R\ge 1}$ is a family of solutions to the equations

$$(-\nabla\cdot a\cdot\nabla - V)v = 0 \quad \text{in } \; \mathcal{C}_U^{(0,R)}. \tag{4.6}$$

Lemma 4.3. *There exist $\gamma > 0$ and $C > 0$ such that for $R \ge 1$ one has*

$$m_{v_{\psi,R}}(R, \Omega) \ge CR^\gamma.$$

Proof. Let w_0 be a quasiconstant solution to (4.5) given by Lemma 4.1. One can check by direct computation (see [7, Lemma 3.4]), that

$$w_R := \frac{w_{\psi,R}}{w_0}$$

is a solution to the equation

$$-\nabla\cdot A\cdot\nabla w = 0 \quad \text{in } \; \mathcal{C}_U^{(0,R)}, \tag{4.7}$$

where $A := w_0^2 a$. Clearly the matrix A is uniformly elliptic with an ellipticity constant $\nu(A) > 0$.

Applying the scaling $y = x/R$ to (4.7) we see that the function $\hat{w}_R(y) = w_R(Ry)$ solves the equation

$$-\nabla\cdot \hat{A}_R \cdot \nabla \hat{w}_R = 0 \quad \text{in } \mathcal{C}_U^{(0,1)},$$

where the matrix $\hat{A}_R(y) = A(Ry)$ is uniformly elliptic with the same ellipticity constant $\nu = \nu(A)$.

Observe that $\partial\mathcal{C}_U^{(0,1)}$ satisfies the exterior cone condition. In particular, every boundary point of $\partial\mathcal{C}_U^{(0,1)}$ is regular. Thus, by the boundary regularity result [3, Theorem 8.27] applied at the vertex $x = 0$ we conclude that there exist $\gamma > 0$ and $C_0 > 0$ such that

$$\mathrm{osc}_{\mathcal{C}_U^{(0,1/R)}} \hat{w}_R(y) \le CR^{-\gamma} \sup_{\mathcal{C}_U^{(0,1/2)}} \hat{w}_R(y) \le C_0 M_\infty R^{-\gamma}.$$

The constants $\gamma > 0$ and $C_0 > 0$ depend only on the ellipticity constant $\nu(A)$ and do not depend on R.

By the same regularity result [3, Theorem 8.27] applied at Ω (considered as a portion of the boundary of $\mathcal{C}_U^{(0,1)}$ we conclude that for some $\delta \in (0, 1/2)$ there exist $C_1 > 0$ and $\gamma_1 > 0$ such that

$$\mathrm{osc}_{\mathcal{C}_\Omega^{(1-\delta,1)}} \hat{w}_R(y) \le C_1 \delta^{\gamma_1} \sup_{\mathcal{C}_\Omega^{(0,1/2)}} \hat{w}_R(y) \le C_1 M_\infty \delta^{\gamma_1}.$$

Here $\gamma_1 > 0$ and $C_1 > 0$ depend only on $\nu(A)$ and do not depend on R. Hence the strong Harnack inequality implies that there exists a constant $M_1 > 0$ such that

$$\inf_{\mathcal{C}_\Omega^{(1/2,1)}} \hat{w}_R(y) \ge M_1.$$

Applying the inverse rescaling $x = Ry$, we conclude that

$$m_{v_{\psi,R}}(R, \Omega) \ge CR^\gamma$$

with some $C > 0$ which is independent of R. □

Lemma 4.4. *There exists a growing solution to equation* (4.1) *that satisfies* (4.3).

Proof. By the Harnack inequality for any compact $K \subset \mathcal{C}_\Omega^{(0,R)}$ such that $K_0 \subset K$ one has

$$\sup_K v_{\psi,R} \le c \inf_K v_{\psi,R} \le c \inf_{K_0} v_{\psi,R} = c,$$

where $c = c(K) > 0$. Let $R_n \to \infty$. By the standard Cacciopoli and diagonalization arguments (see, e.g., [5, Proposition 1.1]) one can construct a function $v_\psi \in H^1_{loc}(\mathcal{C}_\Omega)$ that is a solution to (4.1) in $\mathcal{C}_\Omega$ and satisfies $v_\psi \ge v_{\psi,R_n}$ in $\mathcal{C}_\Omega^{(0,R_n)}$ for each $n \in \mathbb{N}$. Therefore v_ψ is a growing solution to (4.1) in $\mathcal{C}_\Omega$ that obeys (4.3), as required. □

Now we prove that the existence of a growing (super) solution to (4.3) implies a lower bound on the critical exponent $p_*(a, \mathcal{C}_\Omega)$.

Proposition 4.5. $p_*(a, \mathcal{C}_\Omega) \ge 1 - 2/\gamma$, *where* $\gamma > 0$ *is taken from* (4.3).

Proof. Let $w > 0$ be a growing supersolution to (4.1) that satisfies (4.3), as constructed in Lemma 4.4. Fix $p < p_0 = 1 - 2/\gamma$ and set $\delta = p_0 - p$. Then one can choose $\tau = \tau(\delta) > 0$ such that

$$(\tau w)^{p-1} \le \tau^{p-1}(c|x|^\gamma)^{p-1} \le \frac{(c\tau)^{p-1}}{|x|^{2+\delta\gamma}} \le \frac{\epsilon}{|x|^2 \log^2 |x|} \qquad \text{in } \mathcal{C}_\Omega^\rho.$$

Therefore

$$-\nabla\cdot a\cdot\nabla(\tau w)=\frac{\epsilon}{|x|^2\log^2|x|}(\tau w)\geq(\tau w)^{p-1}(\tau w)=(\tau w)^p\quad\text{in}\quad\mathcal{C}^\rho_\Omega,$$

that is $\tau w>0$ is a supersolution to (1.1) in $\mathcal{C}^\rho_\Omega$. □

5 Proof of Theorem 1.4

Using polar coordinates (r,ω), define a Serrin–type operator on $\mathcal{C}_\Omega$ by

$$L_\delta:=-\frac{\partial^2}{\partial r^2}-\frac{N-1}{r}\frac{\partial}{\partial r}-\frac{\delta(r)}{r^2}\Delta_\omega,\tag{5.1}$$

where Δ_ω is the Laplace–Beltrami operator on Ω, and $\delta:\mathbb{R}_+\to\mathbb{R}_+$ is measurable and squeezed between two positive constants. Then L_δ is a divergence type elliptic operator $-\nabla\cdot a_\delta\cdot\nabla$ with a uniformly elliptic matrix $a_\delta(x)$ (see, e.g., [5, 11]). Clearly, if $\delta(r)\equiv 1$ then $L_\delta=-\Delta$.

Proof of Theorem 1.4. Let $\Omega\Subset S^{N-1}$ be a proper subdomain, $\lambda_1=\lambda_1(\Omega)>0$ the principal Dirichlet eigenvalue of $-\Delta_\omega$ on Ω and $\phi_1>0$ the corresponding principal eigenfunction. Given $p\in(-\infty,1)$, set $\beta:=\frac{2}{1-p}$ and consider the operator L_δ with

$$\delta(r)\equiv\frac{\beta(\beta+N-2)}{\lambda_1}.$$

A direct computation shows that

$$w=r^\beta\phi_1(\omega)$$

is a positive solution to $L_\delta w=0$ in $\mathcal{C}_\Omega$. By Proposition 3.2 we conclude that $p_*(a_\delta,\Omega)\leq p$.

Next we show that $p_*(a_\delta,\Omega)\geq p$. To make the arguments more transparent, we make an additional assumption that $\Omega\Subset S^{N-1}$ is smooth. Then for arbitrary $\varepsilon>0$ one can find a proper (smooth) subdomain $\Omega_\varepsilon\Subset S^{N-1}$ such that $\Omega\Subset\Omega_\varepsilon$ and $\lambda_1(\Omega_\varepsilon)\geq\lambda_1(\Omega)-\varepsilon$. Let $\beta:=\frac{2}{1-p}$ and $\gamma_\varepsilon>0$ be the positive root of the quadratic equation

$$-\gamma(\gamma+N-2)+\beta(\beta+N-2)\frac{\lambda_1(\Omega_\varepsilon)}{\lambda_1(\Omega)}=0.$$

Let $\phi_1^{(\epsilon)}>0$ denotes the principal Dirichlet eigenfunction of $-\Delta_\omega$ in Ω_ϵ. Clearly $\gamma_\varepsilon<\beta$ and $\gamma_\varepsilon\to\beta$ as $\varepsilon\to 0$.

A direct computation shows that for all sufficiently small $\varepsilon>0$ the function

$$w_\varepsilon=r^{\gamma_\varepsilon-\varepsilon}\phi_1^{(\varepsilon)}(\omega)$$

is a positive supersolution to the equation

$$(L_\delta-V_{\epsilon_*})\,w=0\quad\text{in}\;\;\mathcal{C}^{\rho_\varepsilon}_{\Omega_\varepsilon}$$

for some $\rho_\varepsilon\gg 1$ (where V_{ϵ_*} is defined in (4.2) and $\epsilon_*>0$ is fixed). Clearly,

$$w_\varepsilon\geq c_\varepsilon r^{\gamma_\varepsilon-\varepsilon}\quad\text{in}\;\;\mathcal{C}^{\rho_\varepsilon}_\Omega.$$

By Proposition 4.5 we conclude that

$$p_*(a_d,\mathcal{C}_\Omega)\geq 1-\frac{2}{\gamma_\varepsilon-\varepsilon}\;\to\;1-\frac{2}{\beta}\quad\text{as}\;\;\varepsilon\to 0,$$

which completes the proof for smooth domains Ω. The proof for the general open subdomains $\Omega\Subset S^{N-1}$ could be carried over following, with minor modifications, the lines of the (rather technical) arguments in [9, Lemma 6.8]. □

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